

Magnetic Relativistic Schrödinger Operators and Imaginary-time Path Integrals *

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Abstract

Three magnetic relativistic Schrödinger operators corresponding to the classical relativistic Hamiltonian symbol with magnetic vector and electric scalar potentials are considered, dependent on how to quantize the kinetic energy term $\sqrt{(\xi - A(x))^2 + m^2}$. We discuss their difference in general and their coincidence in the case of constant magnetic fields, and also study whether they are covariant under gauge transformation. Then results are reviewed on path integral representations for their respective imaginary-time relativistic Schrödinger equations, i.e. heat equations, by means of the probability path space measure related to the Lévy process concerned.

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1 Introduction

We consider the quantized operator $H := H_A + V$ corresponding to the symbol of the classical relativistic Hamiltonian

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d, \quad (1.1)$$

for a *relativistic* spinless particle of mass m under influence of the *magnetic* vector potential $A(x)$ and electric scalar potential $V(x)$ being, respectively, an \mathbf{R}^d -valued function and a real-valued function on space \mathbf{R}^d . This H is effectively used in the situation where one may ignore quantum-field theoretic effects like particles creation and annihilation but should take relativistic effect into consideration. Throughout, the speed c of light and the constant $\hbar := h/2\pi$, the Planck's constant h divided by 2π , are taken to be equal to 1.

If the vector potential $A(x)$ is absent, i.e. $A(x)$ is a constant vector, we can define H as $H = H_0 + V$, where $H_0 := \sqrt{-\Delta + m^2}$ with $-\Delta$ the Laplace operator in \mathbf{R}^d and V is a multiplication operator by the function $V(x)$. We can then realize not only these H_0 and V but also their sum $H_0 + V$ as selfadjoint operators in $L^2(\mathbf{R}^d)$, so long as we consider some class of reasonable scalar potential functions $V(x)$. However, when the vector potential $A(x)$ is present, the definition of H involves some sort of ambiguity. In fact, in the literature there are three kinds of quantum relativistic Hamiltonians dependent on how to quantize the kinetic energy symbol $\sqrt{(\xi - A(x))^2 + m^2}$ to get the first term H_A of H , the kinetic energy operator.

In this article, we will treat these three quantized operators $H^{(1)} = H_A^{(1)} + V$, $H^{(2)} = H_A^{(2)} + V$ and $H^{(3)} = H_A^{(3)} + V$ corresponding to the classical relativistic Hamiltonian symbol (1.1) which have the following kinetic energy parts $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$. At first here, at least in this introduction, we assume for simplicity that $A(x)$ is a *smooth* \mathbf{R}^d -valued function which together with all its derivatives is bounded and that $V(x)$ is a real-valued bounded function.

The first two $H_A^{(1)}$ and $H_A^{(2)}$ are to be defined as pseudo-differential operators through oscillatory integrals. For a function f in $C_0^\infty(\mathbf{R}^d)$ put

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) dy d\xi, \quad (1.2)$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \times \sqrt{\left(\xi - \int_0^1 A((1-\theta)x + \theta y) d\theta\right)^2 + m^2} f(y) dy d\xi. \quad (1.3)$$

The third $H_A^{(3)}$ is defined as the square root of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbf{R}^d)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \quad (1.4)$$

$H_A^{(1)}$ is the so-called Weyl pseudo-differential operator defined with “mid-point prescription” treated in Ichinose–Tamura [ITa-86], Ichinose [I2-88, 3-89, 6-95]. $H_A^{(2)}$ is a modification of $H_A^{(1)}$ given by Iftimie–Măntoiu–Purice [IfMp1-07, 2-08, 3-10] with their

other papers. However, $H_A^{(3)}$ does not seem to be defined as a pseudo-differential operator corresponding to a certain tractable symbol. Indeed, so long as it is defined through Fourier and inverse-Fourier transforms, the candiadte of its symbol will not be $\sqrt{(\xi - A(x))^2 + m^2}$ of (1.1). The last $H^{(3)}$ is used, for instance, to study “stability of matter” in relativistic quantum mechanics in Lieb–Seiringer [LSe-10]. Needles to say, we can show that these three relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ define *selfadjoint* operators in $L^2(\mathbf{R}^d)$.

Then, letting H be one of the magnetic relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ with $H_A^{(1)}$, $H_A^{(2)}$, $H_A^{(3)}$ in (1.2), (1.3), (1.4), consider the following *imaginary-time relativistic Schrödinger equation*, i.e. maybe called *heat equation* for $H - m$:

$$\frac{\partial}{\partial t} u(x, t) = -[H - m]u(x, t), \quad t > 0, \quad x \in \mathbf{R}^d. \quad (1.5)$$

The solution of the Cauchy problem with initial data $u(x, 0) = g(x)$ is given by the semigroup $u(x, t) = (e^{-t[H-m]}g)(x)$. We want to deal with path integral representation for each $e^{-t[H^{(j)}-m]}g$ ($j = 1, 2, 3$). The path integral concerned is connected with the *Lévy process* (e.g. [IkW2-81/89], [Sa2-99], [Ap-04/09]) on the space $D_x := D_x([0, \infty) \rightarrow \mathbf{R}^d)$ dependent on each $x \in \mathbf{R}^d$, of the “*càdlàg* paths”, i.e. right-continuous paths $X : [0, \infty) \ni s \mapsto X(s) \in \mathbf{R}^d$ having left-hand limits and satisfying $X(0) = x$. As our probability space (Ω, P) which is a pair of space Ω and probability P , though here and below not mentioning a σ -algebra on Ω , we take a pair (D_x, λ_x) of the path space D_x and the associated path space measure λ_x on D_x , a probability measure whose characteristic function is given by

$$e^{-t[\sqrt{\xi^2+m^2}-m]} = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{i(X(t)-x) \cdot \xi} d\lambda_x(X), \quad t \geq 0, \quad \xi \in \mathbf{R}^d. \quad (1.6)$$

We will suppress use of the word “random variable” $\omega \in \Omega$.

The aim of this article is to make review, mainly from our results, first on some properties of these three magnetic Schrödinger operators as selfadjoint operators, for instance, that they are different in general from one another but to coincide when the vector potential $A(x)$ is linear in x , in particular, in the case of constant magnetic fields, and bounded from below by the same greatest lower bound, with study of whether they are gauge-covariant, mainly based on [I2-88, 3-89, 4-92, 8-12], [ITs1-92], [IIw-95], and next on Feynman–Kac–Itô-type path integral representations for their respective *imaginary-time* unitary groups, i.e. *real-time* semigroups mainly based on [ITa-86], [I7-95], [HILo1-12, 2-12]. It will be of some interest to collect them in one place to observe how they look like and different, though all the three are bascally connected with the Lévy process.

In Section 2 we give precise definition of the three magnetic relativistic Schrödinger operators and in Section 3 more general definition, studying their properties. In Section 4 path integral representaions for the semigroups for these three magnetic relativistic Schrödinger operators are given accompanied with arguments about heuristic derivation. At the end a summary is given so as to be able to compare the three path integral formulas obtained.

The content of this article is an expanded version of the lecture with almost the same title given by the author at the International Conference on “Partial Differential Equations and Spectral Theory” organized by M. Demuth, B.-W. Schulze and I. Witt, in Goslar, Germany, August 31–September 6, 2008. A brief note with condensed

content on the subject with sketch of proofs also is written in [I9-12] and a rather informal introductory paper of expository character in [I10-12]. For one of the recent references on the related subjects we refer to [LoHB-11]. I hope the present work will give a little more extensive survey to give reviews.

2 Three magnetic relativistic Schrödinger operators

In Section 1, we introduced, though rather roughly, the three magnetic relativistic Schrödinger operators $H^{(1)} = H_A^{(1)} + V$, $H^{(2)} = H_A^{(2)} + V$, $H^{(3)} = H_A^{(3)} + V$ corresponding to the classical relativistic Hamiltonian symbol (1.1). In this Section we are going to give more unambiguous definitions of them and study their properties. The difference lies in how to define their first terms on the right, kinetic energy operators $H_A^{(1)}$, $H_A^{(2)}$, $H_A^{(3)}$, corresponding to the part $\sqrt{(\xi - A(x))^2 + m^2}$ of the symbol (1.1).

2.1 Their definition and difference

For simplicity, it is assumed here as in Section 1 that the vector potential $A : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a C^∞ function and the scalar potential $V : \mathbf{R}^d \rightarrow \mathbf{R}$ is a function *bounded below*. The space of the C^∞ functions with compact support and the space of rapidly decreasing C^∞ functions in \mathbf{R}^d are denoted respectively by $C_0^\infty(\mathbf{R}^d)$ and $\mathcal{S}(\mathbf{R}^d)$.

The definition of $H_A^{(1)}$, $H_A^{(2)}$ as pseudo-differential operators in (1.2), (1.3) needs the concept of *oscillatory integrals*. If the symbol $a(\eta, y)$ satisfies for some $m_0 \in \mathbf{Z}$ and $\tau_0 \geq 0$ that for any multi-indices $\alpha := (\alpha_1, \dots, \alpha_d)$, $\beta := (\beta_1, \dots, \beta_d)$ of nonnegative integers there exist constants $C_{\alpha\beta}$ such that

$$|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leq C_{\alpha\beta} (1 + |\eta|^2)^{m_0/2} (1 + |y|^2)^{\tau_0/2},$$

then the oscillatory integral (e.g. [Ku-74, Theorem 6.4, p.47])

$$\text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{-iy \cdot \eta} a(\eta, y) dy d\eta := \lim_{\varepsilon \rightarrow 0+} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{-iy \cdot \eta} \chi(\varepsilon \eta, \varepsilon y) a(\eta, y) dy d\eta \quad (2.1)$$

exists, where $\chi(\eta, y)$ is any cutoff function in $\mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$ such that $\chi(0, 0) = 1$. The existence of the limit on the right-hand side of (2.1) is independent of the choice of cutoff functions χ , and shown by integration by parts as follows. First note that

$$(-i\partial_y)^\beta e^{-iy \cdot \eta} = (-\eta)^\beta e^{-iy \cdot \eta}, \quad (-i\partial_\eta)^\alpha e^{-iy \cdot \eta} = (-y)^\alpha e^{-iy \cdot \eta},$$

so that

$$\langle \eta \rangle^{-2l} \langle -i\partial_y \rangle^{2l} e^{-iy \cdot \eta} = e^{-iy \cdot \eta}, \quad \langle y \rangle^{-2l'} \langle -i\partial_\eta \rangle^{2l'} e^{-iy \cdot \eta} = e^{-iy \cdot \eta},$$

where

$$\langle \eta \rangle = (1 + |\eta|^2)^{1/2}, \quad \langle y \rangle = (1 + |y|^2)^{1/2}, \quad \langle -i\partial_y \rangle^2 = (1 - \Delta_y), \quad \langle -i\partial_\eta \rangle^2 = (1 - \Delta_\eta).$$

Then the above note with integration by parts shows the integral before the limit $\varepsilon \rightarrow 0+$ taken is equal to

$$\begin{aligned} & \int \int e^{-iy \cdot \eta} \langle \eta \rangle^{-2l} \langle -i\partial_y \rangle^{2l} (\chi(\varepsilon \eta, \varepsilon y) a(\eta, y)) dy d\eta \\ &= \int \int e^{-iy \cdot \eta} \langle y \rangle^{-2l'} \langle -i\partial_\eta \rangle^{2l'} [\langle \eta \rangle^{-2l} \langle -i\partial_y \rangle^{2l} (\chi(\varepsilon \eta, \varepsilon y) a(\eta, y))] dy d\eta. \end{aligned}$$

In the integral on the right above, if the positive integers l and l' are so taken that $-2l + m_0 < -d$, $-2l' + \tau_0 < -d$, then $\langle y \rangle^{-2l'} \langle -i\partial_\eta \rangle^{2l'} (\langle \eta \rangle^{-2l} \langle -i\partial_y \rangle^{2l} a(\eta, y))$ becomes integrable on $\mathbf{R}^d \times \mathbf{R}^d$. Therefore taking the limit $\varepsilon \rightarrow 0+$, we see by the Lebesgue dominated convergence theorem this integral converges to

$$\int \int \langle y \rangle^{-2l'} \langle -i\partial_\eta \rangle^{2l'} (\langle \eta \rangle^{-2l} \langle -i\partial_y \rangle^{2l} a(\eta, y)) dy d\eta,$$

which implies existence of the integral $\text{Os-} \int \int e^{-iy \cdot \eta} a(\eta, y) dy d\eta$, showing existence of the oscillatory integral (2.1).

For $H_A^{(1)}$ in (1.2) we have the following proposition.

Proposition 2.1. *Let $m \geq 0$. If $A(x)$ is in $C^\infty(\mathbf{R}^d; \mathbf{R}^d)$ and satisfies for some $\tau \geq 0$ that*

$$|\partial_x^\beta A(x)| \leq C_\beta \langle x \rangle^\tau \quad (2.2)$$

for any multi-indices β with constants C_β , then for f in $C_0^\infty(\mathbf{R}^d)$ the Weyl pseudo-differential operator $H_A^{(1)}$ in (1.2) exists as an oscillatory integral and further is equal to a second expression; namely, one has

$$(H_A^{(1)} f)(x) = \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) dy d\xi, \quad (2.3)$$

$$= \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \left(\xi + A\left(\frac{x+y}{2}\right)\right)} \sqrt{\xi^2 + m^2} f(y) dy d\xi. \quad (2.4)$$

Proof. We give only a sketch of the proof, dividing into the two cases $m > 0$ and $m = 0$, where note that in the former case the symbol $((\xi - A(x))^2 + m^2)^{1/2}$ has no singularity, but in the latter case singularity on the set $\{(\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d; |\xi - A(x)| = 0\}$.

(a) *The case $m > 0$.* First we treat the oscillatory integral (2.3). Note that $|\partial_\xi^\alpha \partial_x^\beta ((\xi - A(x))^2 + m^2)^{1/2}| \leq C_{\alpha\beta} \langle \xi \rangle \langle x \rangle^\tau$, and hence $|\partial_\xi^\alpha \partial_x^\beta [((\xi - A(x))^2 + m^2)^{1/2} f(x)]| \leq C_{\alpha\beta} \langle \xi \rangle \langle x \rangle^\tau$.

Since f is taken from $C_0^\infty(\mathbf{R}^d)$, we may take a cutoff function which is only dependent on the variable ξ but not y , i.e. $\chi \in \mathcal{S}(\mathbf{R}^d)$ with $\chi(0) = 1$. Then we have by integration by parts as seen before this proposition,

$$\begin{aligned} (H_A^{(1)} f)(x) &= \lim_{\varepsilon \rightarrow 0+} \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \chi(\varepsilon \xi) \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) dy d\xi \\ &= \lim_{\varepsilon \rightarrow 0+} \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \\ &\quad \times \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\chi(\varepsilon \xi) \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) \right) dy d\xi. \end{aligned} \quad (2.5)$$

If we take l sufficiently large, $\langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} (\sqrt{(\xi - A(\frac{x+y}{2}))^2 + m^2} f(y))$ becomes integrable on $\mathbf{R}^d \times \mathbf{R}^d$ for fixed x , so that as $\varepsilon \rightarrow 0+$, by the Lebesgue dominated convergence theorem we have

$$\begin{aligned} (H_A^{(1)} f)(x) &= \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \\ &\quad \times \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) \right) dy d\xi. \end{aligned} \quad (2.6)$$

This proves existence of the second oscillatory integral (2.3) for $H_A^{(1)}$ when $m > 0$. Similarly, we can show existence of the second oscillatory integral (2.4). This shows the first part of the proposition.

To show the second part, i.e. coincidence of the two expressions (2.3) and (2.4), first suppose that $A(x)$ is C_0^∞ . In the integral of the second member of (2.5), we make the change of variables: $\xi = \xi' + A(\frac{x+y'}{2})$, $y = y'$, where we note the Jacobian $|\frac{\partial(\xi, y)}{\partial(\xi', y')}| = 1$. Then it (= the right-hand side of (2.5)) is equal, with ξ' , y' rewritten as ξ , y again, to

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0+} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + A(\frac{x+y}{2}))} \chi\left(\varepsilon\left(\xi + A\left(\frac{x+y}{2}\right)\right)\right) \sqrt{\xi^2 + m^2} f(y) dy d\xi \\ &= \lim_{\varepsilon \rightarrow 0+} \int \int e^{i(x-y) \cdot \xi} \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left[\chi\left(\varepsilon\left(\xi + A\left(\frac{x+y}{2}\right)\right)\right) \right. \\ & \quad \left. \times \sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) \right] dy d\xi, \end{aligned}$$

where we have integrated by parts when passing from the left-hand side to the right. Note that, since the factor $\langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) \right)$ in the integrand on the right-hand side is integrable on $\mathbf{R}^d \times \mathbf{R}^d$ for l sufficiently large with x fixed, we can take the limit $\varepsilon \rightarrow 0+$. So the right-hand side turns out to be equal to

$$\begin{aligned} & \int \int e^{i(x-y) \cdot \xi} \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) \right) dy d\xi \\ &= \lim_{\varepsilon \rightarrow 0+} \int \int e^{i(x-y) \cdot \xi} \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\chi(\varepsilon\xi) \sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) \right) dy d\xi \\ &= \lim_{\varepsilon \rightarrow 0+} \int \int e^{i(x-y) \cdot \xi} \chi(\varepsilon\xi) \sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) dy d\xi \\ &=: \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + A(\frac{x+y}{2}))} \sqrt{\xi^2 + m^2} f(y) dy d\xi. \end{aligned} \tag{2.7}$$

Here the second equality is due to integration by parts. This shows coincidence of (2.3) and (2.4) for $A \in C_0^\infty(\mathbf{R}^d; \mathbf{R}^d)$.

Next, we come to the general case where $A \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$ satisfies (2.2). For this $A(x)$ there exists a sequence $\{A_n(x)\}_{n=1}^\infty \subset C_0^\infty(\mathbf{R}^d; \mathbf{R}^d)$ which converges to $A(x)$ in the topology of $C^\infty(\mathbf{R}^d; \mathbf{R}^d)$, i.e. the $A_n(x)$, together with all their derivatives, converge to $A(x)$ as $n \rightarrow \infty$ uniformly on every compact subset of \mathbf{R}^d . Then we have seen above the coincidence of the two expressions (2.3) and (2.4) for the Weyl pseudo-differential operator $H_{A_n}^{(1)}$ corresponding to the symbol $((\xi - A_n(y))^2 + m^2)^{1/2}$. Therefore, observing (2.6) and (2.7) with A_n in place of A , we obtain

$$\begin{aligned} & \int \int e^{i(x-y) \cdot \xi} \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\sqrt{\left(\xi - A_n\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) \right) dy d\xi \\ &= (2\pi)^d (H_{A_n}^{(1)} f)(x) \\ &= \int \int e^{i(x-y) \cdot \xi} \langle \xi \rangle^{-2l} \langle -i\partial_y \rangle^{2l} \left(\sqrt{\xi^2 + m^2} e^{i(x-y) \cdot A_n(\frac{x+y}{2})} f(y) \right) dy d\xi. \end{aligned}$$

Then we see with the Lebesgue convergence theorem that as $n \rightarrow \infty$, the first member and the third converge to (2.3) and (2.4), respectively, showing the coincidence of (2.3) and (2.4) in the general case.

(b) *The case $m = 0$.* For our cutoff function $\chi(\xi)$, take it from $C_0^\infty(\mathbf{R}^d)$ and require further rotational symmetricity such that $0 \leq \chi(\xi) \leq 1$ for all $\xi \in \mathbf{R}^d$ and $\chi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$; $= 0$ for $|\xi| \geq 1$. Put $\chi_n(\xi) = \chi(\xi/n)$ for positive integer n . Then split the symbol $|\xi - A(x)|$ into a sum of two terms: $|\xi - A(x)| = h_1(\xi, x) + h_2(\xi, x)$,

$$h_1(\xi, x) = \chi_n(\xi - A(x))|\xi - A(x)|, \quad h_2(\xi, x) = [1 - \chi_n(\xi - A(x))]| \xi - A(x)|.$$

Although then the symbol $h_1(\xi, x)$ has singularity, the corresponding Weyl pseudo-differential operator can define a bounded operator on $L^2(\mathbf{R}^d)$ well, and so there is no problem. The one corresponding to the symbol $h_2(\xi, x)$, which has no more singularity, is a pseudo-differential operator defined by oscillatory integral, to which the method in the case (a) above will apply. This ends the proof of Proposition 2.1. \square

For $H_A^{(2)}$ in (1.3), we can show the following proposition in the same way as Proposition 2.1.

Proposition 2.2. *Under the same hypothesis for $A(x)$ as in Proposition 2.1, for f in $C_0^\infty(\mathbf{R}^d)$ the pseudo-differential operator $H_A^{(2)}$ in (1.3) exists as an oscillatory integral and further is equal to a second expression; namely, one has*

$$(H_A^{(2)} f)(x) = \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \times \sqrt{\left(\xi - \int_0^1 A((1-\theta)x + \theta y) d\theta \right)^2 + m^2} f(y) dy d\xi, \quad (2.8)$$

$$= \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \left(\xi + \int_0^1 A((1-\theta)x + \theta y) d\theta \right)} \times \sqrt{\xi^2 + m^2} f(y) dy d\xi. \quad (2.9)$$

In the following, let us gather here, for our three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$, $H_A^{(3)}$, all their definition for up to the present. The most general definition will be given in Section 3.

Definition 2.3. For $A \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$ satisfying condition (2.2), $H_A^{(1)}$ is defined as the pseudo-differential operators (2.3) and/or (2.4).

Definition 2.4. For $A \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$ satisfying condition (2.2), $H_A^{(2)}$ is defined as the pseudo-differential operators (2.8) and/or (2.9).

The definition of $H_A^{(3)}$ encounters a situation totally different from the previous $H_A^{(1)}$ and $H_A^{(2)}$ case. We need nonnegative selfadjointness of the operator $(-\nabla - A(x))^2$ in $L^2(\mathbf{R}^d)$, which can be considered as nothing but the nonrelativistic magnetic Schrödinger operator for a particle with mass $m = \frac{1}{2}$. Of course, if $A \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$, $(-\nabla - A(x))^2$ becomes a nonnegative, selfadjoint operator. However, more generally, it is shown by Kato and Simon (see [CFGKS, pp.8–10]) that if $A \in L_{\text{loc}}^2(\mathbf{R}^d; \mathbf{R}^d)$, $C_0^\infty(\mathbf{R}^d)$ is a form core for the quadratic form for $(-\nabla - A(x))^2$ in $L^2(\mathbf{R}^d)$, so that by the well known argument (e.g. [Kat-76, VI, §2, Theorems 2.1, 2.6, pp.322–323]) there exists a unique nonnegative, selfadjoint operator in $L^2(\mathbf{R}^d)$ associated with this quadratic form with form domain $\{u \in L^2(\mathbf{R}^d); (-\nabla - A(x))u \in L^2(\mathbf{R}^d)\}$. One may take it as $(-\nabla - A(x))^2$. Then its square root $\sqrt{(-i\nabla - A(x))^2 + m^2}$ exists as a nonnegative, selfadjoint operator in $L^2(\mathbf{R}^d)$. This give the following definition for $H_A^{(3)}$.

Definition 2.5. If For $A \in L^2_{\text{loc}}(\mathbf{R}^d; \mathbf{R}^d)$, $H_A^{(3)}$ is defined as the square root :

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2} \quad (2.10)$$

of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbf{R}^d)$.

We note that this $H_A^{(3)}$ does not seem to be defined as a pseudo-differential operator corresponding to a certain tractable symbol. So long as pseudo-differential operators are defined through Fourier and inverse-Fourier transforms, the candidate of its symbol will not be $\sqrt{(\xi - A(x))^2 + m^2}$. The $H_A^{(3)}$ is used, for instance, to study “stability of matter” in relativistic quantum mechanics in Lieb–Seiringer [LSei-10]. An kinetic energy inequality in the presence of the vector potential for the relativistic Schrödinger operators $H_A^{(1)}$ and $H_A^{(3)}$ as well as the nonrelativistic Schrödinger operator $(2m)^{-1}(-i\nabla - A(x))^2$ was given in [I6-93].

Needles to say, we can show that not only $H_A^{(3)}$ but also $H_A^{(1)}$ and $H_A^{(2)}$ define selfadjoint operators in $L^2(\mathbf{R}^d)$. They are in general different from one another but coincide with one another if $A(x)$ is linear in x . We observe these facts in the following.

Proposition 2.6. $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are in general different.

Proof. First, one has $H_A^{(1)} \neq H_A^{(2)}$ for general vector potentials A , because we have

$$A\left(\frac{x+y}{2}\right) \neq \int_0^1 A(x + \theta(y-x))d\theta.$$

Indeed, for instance, for $d = 3$, taking $A(x) \equiv (A_1(x), A_2(x), A_3(x)) = (0, 0, x_3^2)$, we have

$$\begin{aligned} \int_0^1 A_3(x + \theta(y-x))d\theta &= \int_0^1 (x_3 + \theta(y_3 - x_3))^2 d\theta = \frac{x_3^2 + x_3 y_3 + y_3^2}{3} \\ &\neq \left(\frac{x_3 + y_3}{2}\right)^2 = A_3\left(\frac{x+y}{2}\right). \end{aligned}$$

Next, to see that $H_A^{(1)} \neq H_A^{(3)}$ and $H_A^{(2)} \neq H_A^{(3)}$, one needs to show (e.g. [Ho-85, Section 18.5, p.150–152]), for some $g \in C_0^\infty(\mathbf{R}^3)$, respectively, that

$$\begin{aligned} \frac{1}{(2\pi)^6} \int \int \int \int e^{i(x-z) \cdot \zeta + i(z-y) \cdot \eta} \left[\left(\zeta - A\left(\frac{x+z}{2}\right) \right)^2 + m^2 \right]^{1/2} \\ \times \left[\left(\eta - A\left(\frac{z+y}{2}\right) \right)^2 + m^2 \right]^{1/2} g(y) dz d\zeta dy d\eta \neq [(-\nabla - A(x))^2 + m^2] g(x), \end{aligned}$$

and that

$$\begin{aligned} \frac{1}{(2\pi)^6} \int \int \int \int e^{i(x-z) \cdot \zeta + i(z-y) \cdot \eta} \left[\left(\zeta - \int_0^1 A(x + \theta(z-x))d\theta \right)^2 + m^2 \right]^{1/2} \\ \times \left[\left(\eta - \int_0^1 A(z + \theta(y-z))d\theta \right)^2 + m^2 \right]^{1/2} g(y) dz d\zeta dy d\eta \neq [(-\nabla - A(x))^2 + m^2] g(x). \end{aligned}$$

Here the integrals with respect to the space variables above and below are oscillatory integrals.

The former for $H_A^{(1)}$ was shown by Umeda–Nagase [UNa-93, Section 7, p.851]. Indeed, putting $p(x, \xi) := \sqrt{(\xi - A(x))^2 + m^2}$, they verified the (Weyl) symbol $(p \circ p)(x, \xi)$ of $(H_A^{(1)})^2$ satisfy (see [UNa-93, Lemma 6.3, p.846]; [Ho-pp.151–152]) that

$$\begin{aligned} (p \circ p)(x, \xi) &= \frac{1}{(2\pi)^6} \int \int \int \int e^{-i(y \cdot \eta + z \cdot \zeta)} p(x + \frac{z}{2}, \xi - \eta) p(x - \frac{y}{2}, \xi - \eta) dy d\eta dz d\zeta \\ &= \frac{1}{(2\pi)^6} \int \int \int \int e^{i(x-z+y/2) \cdot (\zeta - \xi) + i(z-x+y/2) \cdot (\eta - \xi)} \\ &\quad \times p(\frac{x+z+y/2}{2}, \zeta) p(\frac{x+z-y/2}{2}, \eta) dz d\zeta dy d\eta \\ &\neq (\xi - A(x))^2 + m^2. \end{aligned}$$

The latter for $H_A^{(2)}$ will be shown in a similar way. \square

Theorem 2.7. *If $A(x)$ is linear in x , i.e. if $A(x) = \dot{A} \cdot x$ with \dot{A} being any $d \times d$ real symmetric constant matrix, then $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ coincide. In particular, this holds for constant magnetic fields with $d = 3$, i.e. when $\nabla \times A(x)$ is constant.*

Proof. Suppose $A(x) = \dot{A} \cdot x$. First, we see that $H_A^{(1)} = H_A^{(2)}$ because we have

$$\begin{aligned} \int_0^1 A(\theta x + (1-\theta)y) d\theta &= \int_0^1 \dot{A} \cdot (\theta x + (1-\theta)y) d\theta = \int_0^1 \dot{A} \cdot (y + \theta(x-y)) d\theta \\ &= \dot{A} \cdot \frac{x+y}{2} = A(\frac{x+y}{2}), \end{aligned}$$

which turns out to be “midpoint prescription” to yield the Weyl quantization.

To see that they also coincide with $H_A^{(3)}$, we need to show that $(H_A^{(1)})^2 = (-i\nabla - A(x))^2 + m^2$. To do so, let $f \in C_0^\infty(\mathbf{R}^d)$ and note $(\dot{A})^T = \dot{A}$, then we have, with integrals as oscillatory integrals,

$$\begin{aligned} &((H_A^{(1)})^2 f)(x) \\ &= \frac{1}{(2\pi)^{2d}} \int \int \int \int e^{i(x-z) \cdot (\eta + \dot{A} \frac{x+z}{2}) + i(z-y) \cdot (\xi + \dot{A} \frac{z+y}{2})} \sqrt{\eta^2 + m^2} \sqrt{\xi^2 + m^2} f(y) dz d\eta dy d\xi \\ &= \frac{1}{(2\pi)^{2d}} \int \int \int \int e^{iz \cdot (-\eta + \xi)} e^{i[x \cdot (\eta + \dot{A} \frac{x}{2}) - y \cdot (\xi + \dot{A} \frac{y}{2})]} \sqrt{\eta^2 + m^2} \sqrt{\xi^2 + m^2} f(y) dz d\eta dy d\xi \\ &= \frac{1}{(2\pi)^d} \int \int \int \delta(-\eta + \xi) e^{i[x \cdot (\eta + \dot{A} \frac{x}{2}) - y \cdot (\xi + \dot{A} \frac{y}{2})]} \sqrt{\eta^2 + m^2} \sqrt{\xi^2 + m^2} f(y) d\eta dy d\xi. \end{aligned}$$

Hence

$$\begin{aligned} ((H_A^{(1)})^2 f)(x) &= \frac{1}{(2\pi)^d} \int \int e^{i(x-y) \cdot \xi} e^{i\frac{1}{2}(x \cdot \dot{A} x - y \cdot \dot{A} y)} (\xi^2 + m^2) f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \int \int e^{i(x-y) \cdot (\xi + \dot{A} \frac{x+y}{2})} (\xi^2 + m^2) f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \int \int e^{i(x-y) \cdot (\xi + A(\frac{x+y}{2}))} (\xi^2 + m^2) f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \int \int e^{i(x-y) \cdot \xi} [(\xi - A(\frac{x+y}{2}))^2 + m^2] f(y) dy d\xi. \end{aligned}$$

The last equality is due to the fact that symbol $(\xi - A(x))^2 + m^2$ is polynomial of ξ , so that the corresponding Weyl pseudo-differential operator is equal to $(-i\nabla - A(x))^2 + m^2$. \square

2.2 Gauge-covariant or not

Among these three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$, the Weyl quantized one like $H_A^{(1)}$ (in general, the Weyl pseudo-differential operator) is *compatible well with path integral* (e.g. Mizrahi [M-78]). But the pity is that, for general vector potential $A(x)$, $H_A^{(1)}$ (and so $H^{(1)}$) is not generally covariant under gauge transformation, namely, there exists a real-valued function $\varphi(x)$ for which it fails to hold that $H_{A+\nabla\varphi}^{(1)} = e^{i\varphi} H_A^{(1)} e^{-i\varphi}$.

However, $H_A^{(2)}$ (and so $H^{(2)}$) and $H_A^{(3)}$ (and so $H^{(3)}$) are gauge-covariant, though these three are not in general equal as seen in Proposition 2.6. The gauge-covariance of the modified $H_A^{(2)}$ in contrast to $H_A^{(1)}$ in Ichinose–Tamura [ITa-86] was emphasized in Iftimie–Măntoiu–Purice [IfMP1-07, 2-08, 3-10]. There, in particular, in [IfMP1-07], they also compared our three magnetic Schrödinger operators to observe the following facts.

Proposition 2.8. $H_A^{(2)}$ and $H_A^{(3)}$ are covariant under gauge transformation, i.e. it holds for $j = 2, 3$ that $H_{A+\nabla\varphi}^{(j)} = e^{i\varphi} H_A^{(j)} e^{-i\varphi}$ for every $\varphi \in \mathcal{S}(\mathbf{R}^d)$. But $H_A^{(1)}$ is in general not covariant under gauge transformation.

Poof. First, to see the assertion for $H_A^{(3)} = \sqrt{(-i\nabla - A(x))^2 + m^2}$, put $K_A = (-i\nabla - A(x))^2 + m^2$, so that $H_A^{(3)} = K_A^{1/2}$. As $(-i\nabla - A(x))^2$ is a nonrelativistic magnetic Schrödinger operator with mass $\frac{1}{2}$, being a nonnegative selfadjoint operator on $L^2(\mathbf{R}^d)$, and gauge-covariant, so is K_A . Therefore it satisfies $K_{A+\nabla\varphi} = e^{i\varphi} K_A e^{-i\varphi}$ for every $\varphi(x)$. It follows that $(K_{A+\nabla\varphi}^{1/2})^2 = (e^{i\varphi} K_A^{1/2} e^{-i\varphi})(e^{i\varphi} K_A^{1/2} e^{-i\varphi})$, whence $K_{A+\nabla\varphi}^{1/2} = e^{i\varphi} K_A^{1/2} e^{-i\varphi}$, because $e^{i\varphi} K_A^{1/2} e^{-i\varphi}$ is also nonnegative selfadjoint. This means that $H_{A+\nabla\varphi}^{(3)} = e^{i\varphi} H_A^{(3)} e^{-i\varphi}$, i.e. $H_A^{(3)}$ is gauge-covariant.

Next, for $H_A^{(2)}$, by mean-value theorem

$$\begin{aligned} \varphi(y) - \varphi(x) &= \int_0^1 (y - x) \cdot (\nabla\varphi)(x + \theta(y - x)) d\theta \\ &= - \int_0^1 (x - y) \cdot (\nabla\varphi)((1 - \theta)x + \theta y) d\theta. \end{aligned}$$

Hence

$$\begin{aligned} &(H_A^{(2)} e^{-i\varphi} f)(x) \\ &= \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + \int_0^1 A((1-\theta)x + \theta y) d\theta)} \\ &\quad \times \sqrt{\xi^2 + m^2} e^{-i\varphi(x) + i \int_0^1 (x-y) \cdot (\nabla\varphi)((1-\theta)x + \theta y) d\theta} f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} e^{-i\varphi(x)} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + \int_0^1 (A + \nabla\varphi)((1-\theta)x + \theta y) d\theta)} \sqrt{\xi^2 + m^2} f(y) dy d\xi \\ &= e^{-i\varphi(x)} (H_{A+\nabla\varphi}^{(2)} f)(x). \end{aligned}$$

Finally, we show non-gauge-invariance of $H_A^{(1)}$. To this end, we are going to use a second expression for $H_A^{(1)}$ as an integral operator to be given in the next section,

(3.7) in Definition 3.7. Then we show that it does not hold for all φ that $H_{A+\nabla\varphi}^{(1)} = e^{i\varphi} H_A^{(1)} e^{-i\varphi}$ or that, taking $A \equiv 0$. Indeed, suppose that

$$\begin{aligned} & \text{p.v.} \int_{|y|>0} [e^{-iy \cdot (\nabla\varphi)(x+\frac{y}{2})} f(x+y) - f(x)] n(dy) \\ &= e^{i\varphi(x)} \text{p.v.} \int_{|y|>0} [(e^{-i\varphi} f)(x+y) - (e^{-i\varphi} f)(x)] n(dy) \\ &\equiv \text{p.v.} \int_{|y|>0} [e^{-i(\varphi(x+y)-\varphi(x))} f(x+y) - f(x)] n(dy). \end{aligned}$$

However, the second equality cannot hold, because it does not hold for all φ that $\varphi(x+y) - \varphi(x) = y \cdot (\nabla\varphi)(x + \frac{y}{2})$. \square

3 More general definition of magnetic relativistic Schrödinger operators and their selfadjointness

In this section, we want to give the most general definition of $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$, which do *not* appeal to the pseudo-differential operators.

3.1 The most general definition of $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$

First we concern $H_A^{(1)}$ and $H_A^{(2)}$. The starting point is the *Lévy-Khinchin formula* for the conditionally negative definite function $\sqrt{\xi^2 + m^2} - m$, which has an integral representation with a σ -finite measure $n(dy)$ on $\mathbf{R}^d \setminus \{0\}$, called *Lévy measure*, which satisfies $\int_{|y|>0} \frac{|y|^2}{1+|y|^2} n(dy) < \infty$:

$$\begin{aligned} \sqrt{\xi^2 + m^2} - m &= - \int_{\{|y|>0\}} [e^{iy \cdot \xi} - 1 - iy \cdot \xi I_{\{|y|<1\}}] n(dy) \\ &= - \lim_{r \rightarrow 0+} \int_{|y| \geq r} [e^{iy \cdot \xi} - 1] n(dy) \equiv -\text{p.v.} \int_{|y|>0} [e^{iy \cdot \xi} - 1] n(dy). \end{aligned} \quad (3.1)$$

Here $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y| < 1\}$ in \mathbf{R}^d , i.e. $I_{\{|y|<1\}}(z) = 1$, if $|z| < 1$, and $= 0$, if $|z| \geq 1$. Though the Lévy measure $n(dy)$ is m -dependent, we will suppress explicit indication of m -dependence so as to make the notation simpler. $n(dy)$ has density so that $n(dy) = n(y)dy$. The density function $n(y)$ is given by

$$n(y) = \begin{cases} 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{K_{(d+1)/2}(m|y|)}{|y|^{(d+1)/2}}, & m > 0, \\ \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \frac{1}{|y|^{d+1}}, & m = 0, \end{cases} \quad (3.2)$$

where $\Gamma(\tau)$ is the gamma function, and $K_\nu(\tau)$ the modified Bessel function of the third kind of order ν , which satisfies $0 < K_\nu(\tau) \leq C[\tau^{-\nu} \vee \tau^{-1/2}]e^{-\tau}$, $\tau > 0$ with a constant $C > 0$.

To get (3.2) recall (e.g. [ITa-86, Eq.(4.2), p.244]) the operator $e^{-t[\sqrt{-\Delta+m^2}-m]}$ has integral kernel $k_0(x-y, t)$, where

$$k_0(y, t) = \begin{cases} 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{te^{mt} K_{(d+1)/2}(m(|y|^2+t^2)^{1/2})}{(|y|^2+t^2)^{(d+1)/4}}, & m > 0, \\ \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \frac{t}{(|y|^2+t^2)^{(d+1)/2}}, & m = 0, \end{cases} \quad (3.3)$$

und use that fact (e.g. [IkW1-62, Example 1]) that $\frac{k_0(y,t)}{t}dy$ converges to $n(dy) = n(y)dy$, as measures on \mathbf{R}^d , as $t \rightarrow 0+$. Note that the both the expressions on the right-hand side of $n(y)$ in (3.2) and $k_0(y,t)$ in (3.3) are continuously connected as $m \rightarrow 0+$, because $K_\nu(\tau) = \frac{\Gamma(\nu)}{2} \left(\frac{2}{\tau}\right)^\nu (1 + o(1))$ as $\tau \rightarrow 0+$.

We shall denote by $H_0 \equiv \sqrt{-\Delta + m^2}$ not only the linear operator of $L^2(\mathbf{R}^d)$ into itself with domain $H^1(\mathbf{R}^d)$ but also the linear map $\mathcal{F}^{-1}\sqrt{\xi^2 + m^2}\mathcal{F}$ of $\mathcal{S}'(\mathbf{R}^d)$ into itself as well as of the Sobolev space $H^s(\mathbf{R}^d)$ into $H^{s+1}(\mathbf{R}^d)$, where \mathcal{F} and \mathcal{F}^{-1} stand for the Fourier and inverse Fourier transforms. Now for $f \in \mathcal{S}(\mathbf{R}^d)$, put $\hat{f} = \mathcal{F}f$. The inverse Fourier transform of $\hat{f}(\xi)$ multiplied by (3.1) becomes

$$\begin{aligned} (H_0 f)(x) &\equiv (\sqrt{-\Delta + m^2} f)(x) \\ &= m f(x) - \int_{\{|y|>0\}} [f(x+y) - f(x) - I_{\{|y|<1\}} y \cdot \nabla_x f(x)] n(dy). \end{aligned} \quad (3.4)$$

Now to treat $H_A^{(1)}$, let $f \in C_0^\infty(\mathbf{R}^d)$ and consider, for each fixed x , the function

$$f_x : y \mapsto e^{i(x-y) \cdot A \left(\frac{x+y}{2}\right)} f(y),$$

which also belongs to $C_0^\infty(\mathbf{R}^d)$. Replace f in (3.4) by f_x , then we get

$$\begin{aligned} (H_0 f_x)(x) & \\ &= m f(x) - \int_{\{|y|>0\}} [e^{-iy \cdot A(x+\frac{y}{2})} f(x+y) - f(x) - I_{\{|y|<1\}} y \cdot (\nabla_x - iA(x)) f(x)] n(dy). \end{aligned} \quad (3.5)$$

On the other hand, notice that the left-hand side of (3.5) is written by use of Fourier transform

$$\begin{aligned} (H_0 f_x)(x) &= \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\xi^2 + m^2} f_x(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \text{Os-} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \left(\xi + A\left(\frac{x+y}{2}\right)\right)} \sqrt{\xi^2 + m^2} f(y) dy d\xi, \end{aligned}$$

the last member of which is nothing but the second oscillatory expression (2.4) for $(H_A^{(1)} f)(x)$. Thus we have obtained the identity

$$(H_0 f_x)(x) = (H_A^{(1)} f)(x) \quad (3.6)$$

for all $f \in C_0^\infty(\mathbf{R}^d)$. This may be paraphrased “apply $H_A^{(1)}$ to f amounts to be the same thing as apply H_0 to f_x ”. Thus we are lead to a new definition of $H_A^{(1)}$:

$$\begin{aligned} ([H_A^{(1)} - m]f)(x) &:= - \int_{|y|>0} [e^{-iy \cdot A(x+\frac{y}{2})} f(x+y) - f(x) \\ &\quad - I_{\{|y|<1\}} y \cdot (\nabla_x - iA(x)) f(x)] n(dy) \\ &= - \lim_{r \rightarrow 0+} \int_{|y| \geq r} [e^{-iy \cdot A(x+\frac{y}{2})} f(x+y) - f(x)] n(dy) \\ &\equiv - \text{p.v.} \int_{|y|>0} [e^{-iy \cdot A(x+\frac{y}{2})} f(x+y) - f(x)] n(dy). \end{aligned} \quad (3.7)$$

This expression makes sense for $A(x)$ being more general functions than C^∞ . In fact, it can be shown with the Calderon–Zygmund theorem that the singular integral on the right-hand side of (3.7) exists pointwise in a.e. x as well as in the L^2 norm, if $A \in L_{\text{loc}}^{2+\delta}(\mathbf{R}^d; \mathbf{R}^d)$ for some $\delta > 0$ such that $\int_{0 < |y| < 1} |A(x + \frac{y}{2}) - A(x)| |y|^{-d} dy$ is L_{loc}^2 . Note that (3.7) reduces itself to (3.4) if $A(x) \equiv 0$.

For $H_A^{(2)}$, we can do it in the same way. Indeed, take $f \in C_0^\infty(\mathbf{R}^d)$ to consider, for x fixed, the function

$$f_x : y \mapsto e^{i(x-y) \cdot \int_0^1 A((1-\theta)x + \theta y) d\theta} f(y),$$

which belongs to $C_0^\infty(\mathbf{R}^d)$. Replacing f in (3.4) by f_x we obtain the relation $(H_0 f_x)(x) = (H_A^{(2)} f)(x)$, namely, a new definition for $H_A^{(2)}$, as follows:

$$\begin{aligned} ([H_A^{(2)} - m]f)(x) &:= - \int_{|y|>0} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x) \\ &\quad - I_{\{|y|<1\}} y \cdot (\nabla_x - iA(x)) f(x)] n(dy) \\ &:= - \lim_{r \rightarrow 0+} \int_{|y| \geq r} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x)] n(dy) \\ &\equiv - \text{p.v.} \int_{|y|>0} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x)] n(dy). \end{aligned} \quad (3.8)$$

Now we are in a position to consider the most general definition for the first two magnetic relativistic Schrödinger operators $H^{(1)} = H_A^{(1)} + V$ and $H^{(2)} = H_A^{(2)} + V$ with both general vector and scalar potentials $A(x)$ and $V(x)$. Assume that

$$A \in L_{\text{loc}}^{1+\delta}(\mathbf{R}^d) \text{ for some } \delta > 0 \text{ and } V \in L_{\text{loc}}^1(\mathbf{R}^d), \quad V(x) \geq 0 \text{ a.e.}, \quad (3.9)$$

or

$$A \in L_{\text{loc}}^{2+\delta}(\mathbf{R}^d) \text{ for some } \delta > 0 \text{ and } V \in L_{\text{loc}}^2(\mathbf{R}^d), \quad V(x) \geq 0 \text{ a.e.} \quad (3.10)$$

Then, first, if A and V satisfy (3.10), we can see again with the Calderon–Zygmund theorem that the singular integrals on the right-hand side of (3.7) and (3.8) exist pointwise in a.e. x as well as in the L^2 norm.

Next, if A and V satisfy (3.9), multiply (3.7) and (3.8) with $\overline{u(x)}$ and integrate them by dx , then we can reach the following quadratic forms $h^{(1)}$ and $h^{(2)}$, respectively:

$$\begin{aligned} h^{(1)}[u] &\equiv h^{(1)}[u, u] := h_{A,V}^{(1)}[u, u] \\ &= \left(m\|u\|^2 + \frac{1}{2} \int \int_{|x-y|>0} |e^{-i(x-y) \cdot A(\frac{1}{2}(x+y))} u(x) - u(y)|^2 n(x-y) dx dy \right) \\ &\quad + \int V(x) |u(x)|^2 dx =: h_A^{(1)}[u] + h_V^{(1)}[u]; \end{aligned} \quad (3.11)$$

$$\begin{aligned} h^{(2)}[u] &\equiv h^{(2)}[u, u] := h_{A,V}^{(2)}[u, u] \\ &= \left(m\|u\|^2 + \frac{1}{2} \int \int_{|x-y|>0} |e^{-i(x-y) \cdot \int_0^1 A((1-\theta)x + \theta y) d\theta} u(x) - u(y)|^2 n(x-y) dx dy \right) \\ &\quad + \int V(x) |u(x)|^2 dx =: h_A^{(2)}[u] + h_V^{(2)}[u] \end{aligned} \quad (3.12)$$

with form domains $Q(h^{(j)}) = \{u \in L^2(\mathbf{R}^d); h^{(j)}[u] < \infty\}$, $j = 1, 2$. We can see with [Kat-76, VI, §2, Theorems 2.1, 2.6, pp.322–323] that under the assumption (3.9) for

$A(x)$ and $V(x)$, there exist unique nonnegative selfadjoint operators $H^{(j)} = H_A^{(j)} \dot{+} V$ (form sum), $j = 1, 2$, such that $h^{(j)}[u, v] = (H^{(j)}u, v)$ for $u, v \in C_0^\infty(\mathbf{R}^d)$.

We expect that the condition (3.10) (resp. (3.9)) is minimal to assure that $H^{(j)}$ (resp. $h^{(j)}$) defines a linear operator (resp. quadratic form) in $L^2(\mathbf{R}^d)$ with domain (resp. form domain) including $C_0^\infty(\mathbf{R}^d)$, so long as $V(x)$ is nonnegative. Then we can show the following results under the assumptions (3.10) and (3.9).

Theorem 3.1. ([ITs2-93]) (i) If $A(x)$ and $V(x)$ satisfy (3.9), for each $j = 1, 2$, $h^{(j)}$ is a closed form with form domain $Q(h^{(j)})$ including $C_0^\infty(\mathbf{R}^d)$ as a form core, so that the minimal symmetric form $h_{\min}^{(j)}$ defined as the form closure of $h^{(j)}|_{C_0^\infty(\mathbf{R}^d) \times C_0^\infty(\mathbf{R}^d)}$ coincides with $h^{(j)}$. Therefore there exists a unique selfadjoint operator $H^{(j)} = H_A^{(j)} \dot{+} V$ (form sum) with domain $D(H^{(j)})$ corresponding to the form $h^{(j)}$ such that $h^{(j)}[u, v] = (H^{(j)}u, v)$ for $u \in D(H^{(j)})$, $v \in Q(h^{(j)})$.

(ii) If $A(x)$ and $V(x)$ satisfy (3.10), for each $j = 1, 2$, $H^{(j)} = H_A^{(j)} + V$ (operator sum) is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ and the closure of $H^{(j)}$, denoted by the same $H^{(j)}$ again, is bounded from below by m .

The proof of statements in Theorem 3.1 for $H^{(1)}$ and $h^{(1)}$ were first given under less general assumption in [I3-89], [I4-93] and [ITs1-92] and completed as in the present form in [ITs2-93], while that for $H^{(2)}$ and $h^{(2)}$ given in [IfMP1-07, 2-08, 3-10] for vector potentials $A(x)$ which are C^∞ functions of polynomial growth as $|x| \rightarrow \infty$.

Thus we are led to more general definitions, not only for the two magnetic relativistic Schrödinger operators $H_A^{(1)}$ and $H_A^{(2)}$ than the ones in Definitions 2.3 and 2.4, but also for the two general relativistic Schrödinger operators $H^{(1)}$ and $H^{(2)}$ with both vector and scalar potentials $A(x)$ and $V(x)$.

Definition 3.2. If $A(x)$ and $V(x)$ satisfy (3.10), for $j = 1, 2$, $H^{(j)}$ is defined as the closure of the operator sum of the integral operator $H_A^{(j)}$ in (3.7), (3.8) and the potential $V(x)$.

Definition 3.3. If $A(x)$ and $V(x)$ satisfy (3.9), for $j = 1, 2$, $H^{(j)}$ is defined as the selfadjoint operator $H^{(j)} = H_A^{(j)} \dot{+} V$ associated with the closed form $h^{(j)}$ which is the sum of the two closed forms $h_A^{(j)}$ and $h_V^{(j)}$ as in (3.11) and (3.12).

Next, we come to $H_A^{(3)}$. If $0 < \delta < 1$, condition “ $A \in L_{\text{loc}}^{1+\delta}(\mathbf{R}^d; \mathbf{R}^d)$ ” in (3.9) for A is slightly more general than condition “ $A \in L_{\text{loc}}^2(\mathbf{R}^d; \mathbf{R}^d)$ ” used to give the definition for $H_A^{(3)}$ in Definition 2.5. As Theorem 3.1 (ii) says that when $A \in L_{\text{loc}}^{1+\delta}(\mathbf{R}^d; \mathbf{R}^d)$, $(-i\nabla - A(x))^2 + m^2$ can define a nonnegative selfadjoint operator in $L^2(\mathbf{R}^d)$, so we are led to the following more general definition than the one in Definition 2.5.

Definition 3.4. If $A(x)$ and $V(x)$ satisfy (3.9), $H^{(3)}$ is defined in $L^2(\mathbf{R}^d)$ as the form sum of the square root of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ and V :

$$H^{(3)} := H_A^{(3)} \dot{+} V := \sqrt{(-i\nabla - A(x))^2 + m^2} \dot{+} V \quad (3.13)$$

Thus, with Definitions 3.2, 3.3 and 3.4, we are now given more general definition of the three relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ concerned, corresponding

to the classical relativistic symbol (1.1) with both vector and scalar potentials $A(x)$ and $V(x)$.

It is appropriate here to refer, for comparison, to the corresponding results for the nonrelativistic magnetic Schrödinger operator $H^{NR} := H_A^{NR} + V := \frac{1}{2}(-i\nabla - A(x))^2 + V(x)$. In fact, as already mentioned in Section 2.1, one can realize H^{NR} as a selfadjoint operator defined through the quadratic form with form domain including $C_0^\infty(\mathbf{R}^d)$ as a form core when

$$A \in L_{\text{loc}}^2(\mathbf{R}^d) \text{ and } V \in L_{\text{loc}}^1(\mathbf{R}^d), \quad V(x) \geq 0 \text{ a.e.}, \quad (3.14)$$

which was proved by Kato and Simon (see [CFKS-87, pp.8–10]). This was also proved by Leinfelder-Simader [LeSi-81], who further gave a definitive result that it is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ when

$$A \in L_{\text{loc}}^4(\mathbf{R}^d), \quad \text{div} A \in L_{\text{loc}}^2(\mathbf{R}^d) \text{ and } V \in L_{\text{loc}}^2(\mathbf{R}^d), \quad V(x) \geq 0 \text{ a.e.} \quad (3.15)$$

The proof of Theorem 3.1 will be carried out by mimicking the arguments used by Leinfelder-Simader [LeSi-81]. First the statement (i) is proved. Then the idea of proof of the statement (ii) consists in showing that, when $A(x)$ and $V(x)$ satisfy (3.10), $C_0^\infty(\mathbf{R}^d)$ is also an operator core of the selfadjoint operator $H^{(1)}$ obtained through the form $h^{(1)}$ in the statement (i). We refer the details of the proof to Ichinose-Tsuchida [ITs2-93].

Remarks. 1°. For the scalar potential $V \in L_{\text{loc}}^2(\mathbf{R}^d)$ having negative part: $V(x) = V_+(x) - V_-(x)$ with $V_\pm(x) \geq 0$ and $V_+(x)V_-(x) = 0$ a.e., it will be possible to show the theorem, if $V_-(x)$ is small in a certain sense [i.e. relatively bounded / relatively form-bounded with respect to $\sqrt{-\Delta + m^2}$ or $H_A^{(j)}$ ($j = 1, 2$) with relative bound less than 1], but we content ourselves with such V as in (3.9) and (3.10). The main point of Theorem 3.1 is in treating the Hamiltonian with vector potential $A(x)$ as general as possible.

2°. Nagase-Umeda [NaU1-90] proved essential selfadjointness of the Weyl pseudo-differential operator $H_A^{(1)}$ in (2.3).

3°. When $A(x)$ is in $L_{\text{loc}}^{2+\delta}(\mathbf{R}^d; \mathbf{R}^d)$ and

$$\int_{0 < |y| < 1} |y \cdot (A(x + y/2) - A(x))| |y|^{-d} dy \text{ is in } L_{\text{loc}}^2(\mathbf{R}^d), \quad (3.16)$$

it can be shown [ITs1-92] (cf. [I3-89]) that *Kato's inequality* holds for $H_A^{(1)}$ in (3.7): If $u \in L^2(\mathbf{R}^d)$ with $H_A^{(1)}u \in L_{\text{loc}}^1(\mathbf{R}^d)$, then the distributional inequality

$$\text{Re}((\text{sgn } u)[H_A^{(1)} - m]u) \geq [\sqrt{-\Delta + m^2} - m] |u| \quad (3.17)$$

holds, where $(\text{sgn } u)(x) = \overline{u(x)}/|u(x)|$ for $u(x) \neq 0$; $= 0$ for $u(x) = 0$. In particular, if $A(x)$ is Hölder-continuous, then $A(x)$ satisfies condition (3.16). To show (3.17), one has to use the expression (3.7) for $H_A^{(1)}f$ instead of (2.3). For the detail see [ITs1-92, Theorem 3.1] (cf. [I3-89, Theorems 4.1, 5.1]).

In the same way, *Kato's inequality* also for $H_A^{(2)}$ will be shown with the expression (3.8) instead of (2.8): If $u \in L^2(\mathbf{R}^d)$ with $H_A^{(2)}u \in L_{\text{loc}}^1(\mathbf{R}^d)$, then the distributional inequality

$$\text{Re}((\text{sgn } u)[H_A^{(2)} - m]u) \geq [\sqrt{-\Delta + m^2} - m] |u| \quad (3.18)$$

holds. Also for $H_A^{(3)}$, we expect to have a distributional inequality like

$$\operatorname{Re}((\operatorname{sgn} u)[H_A^{(3)} - m]u) \geq [\sqrt{-\Delta + m^2} - m] |u|. \quad (3.19)$$

However, the problem will be open. Although it can be shown [HILo2-12] (cf. [HILo1-12]) that inequality (*Diamagnetic inequality*)

$$(f, e^{-t[H_A^{(3)} - m]} f) \leq (|f|, e^{-t[\sqrt{-\Delta + m^2} - m]} |f|) \quad (3.20)$$

holds for all $f \in L^2(\mathbf{R}^d)$, which (cf. [S1-77], [HeScUh-77]) is equivalent to an abstract version of “Kato’s inequality” for $H_A^{(3)}$, the distributional version (3.20) is a stronger assertion. Here and throughout this article, (\cdot, \cdot) in (3.20) is the *physicist’s inner product* (f, g) of the Hilbert space $L^2(\mathbf{R}^d)$, which is anti-linear in f and linear in g .

Finally, we are going to see the three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are bounded from below by the *same lower bound*, as in the following theorem.

Theorem 3.5.

$$H_A^{(j)} \geq m, \quad j = 1, 2, 3. \quad (3.21)$$

Proof. First, it is trivial for $H_A^{(3)}$, as also seen from (3.12), for instance. Next to see for $H_A^{(1)}$, take $u \in C_0^\infty(\mathbf{R}^d)$ in Kato’s inequality (3.17) above. Multiply both sides by $|u(x)|$ and integrate them in x , then we have

$$(u, [H_A^{(1)} - m]u) \geq (|u|, [\sqrt{-\Delta + m^2} - m] |u|),$$

where we note that $|u(x)|$ is in the Sobolev space $H^1(\mathbf{R}^d)$, so that the right-hand side above exists finite and nonnegative. So the assertion follows. In the same way it will be shown for $H_A^{(2)}$. \square

Remark. In the above proof, the sharp lower bound (3.21) for $H_A^{(1)}$ and $H_A^{(2)}$ has been obtained with their integral operator expressions (3.7) and (3.8). It does not seem to be obtained by pseudo-differential calculus from their expressions (2.2)/(2.3) and (2.8)/(2.9), but instead then probably only a bound such as $H_A^{(j)} \geq m - \delta$ for some $\delta > 0$ (cf. [Ho-85, Section 18.1]).

3.2 Selfadjointness with negative scalar potentials

We have seen above that our relativistic Schrödinger operators with *nonnegative* scalar potentials assume analogous aspects on selfadjointness problem with the nonrelativistic Schrödinger operator. In this subsection we shall observe that, as for *negative* scalar potentials unbounded at infinity, the former makes a remarkable contrast with the latter, bearing an aspect closer to the Dirac operator (cf. Chernoff [Ch-77]), though the relativistic Schrödinger equation $i \frac{\partial}{\partial t} \varphi(x, t) = [H_0 - m] \varphi(x, t)$ has not the finite propagation property.

For comparison, first we refer to the result for the nonrelativistic Schrödinger operator $-\Delta + V(x) + W(x)$ in $L^2(\mathbf{R}^d)$ by Faris–Lavine [FaLa-74] (or [RS-75, Theorem X.38, p.198]): Assume that the real-valued scalar potentials V and W obey the following conditions: Let V be in $L_{\text{loc}}^p(\mathbf{R}^d)$ for some $p \geq (d/2) \vee 2$ ($p > 2$ when $d = 4$), and

let W be in $L^2_{\text{loc}}(\mathbf{R})$ and satisfy $W(x) \geq -c_1|x|^2 - c_2$ for some constants c_1, c_2 , then $-\Delta + V(x) + W(x)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$.

Now we consider the magnetic relativistic Schrödinger operator $H^{(1)} = H_A^{(1)} + V + W$, assuming the following conditions: Let A be in $L^{2+\delta}_{\text{loc}}(\mathbf{R}^d; \mathbf{R}^d)$ for some $\delta > 0$ and satisfies (3.16), and V in $L^2_{\text{loc}}(\mathbf{R}^d)$ which is relatively bounded with respect to $H_0 \equiv \sqrt{-\Delta + m^2}$ with relative bound < 1 . Let W be in $L^2_{\text{loc}}(\mathbf{R}^d)$. Assume further with constants $a \geq 0, 0 < b \leq 1, c \geq 0$ that

$$A(x) \text{ is bounded by a polynomial of } |x| \text{ and } W(x) \geq -c \exp[a|x|^{1-b}], \quad (3.22)$$

or

$$A(x) \text{ is bounded and } W(x) \geq -ce^{a|x|}. \quad (3.23)$$

Note that condition (3.22) or (3.23) allows $W(x)$ to decrease exponentially at infinity with respect to $|x|^{1-b}$ or $|x|$.

Then we have the following result in [Iw-94] and [Iw-95] for the magnetic relativistic Schrödinger operator $H^{(1)} = H_A^{(1)} + V + W$. The former work used, for the operator $H_A^{(1)}$, the pseudo-differential operator expression in Definition 2.3 with (2.3), while the latter the integral operator expression connected with Lévy process in Definition 3.2 with (3.7). The latter result is sharper than the former.

Theorem 3.6. ([Iw-95]) *With the above assumption on $A(x)$, $V(x)$ and $W(x)$, $H^{(1)} := H_A^{(1)} + V + W$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$.*

Example. For $0 \leq Z < (d-2)/2, d \geq 3$, the operator $H_A^{(1)} - Z/|x| + W(x)$ with Hölder-continuous $A(x)$ and locally square-integrable $W(x)$ satisfying (3.22) or (3.23) is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$.

Proof of Theorem 3.6. (Sketch) First, using that V is H_0 -bounded with relative bound < 1 , we show that $H_A^{(1)} + V$ is a symmetric operator with domain $C_0^\infty(\mathbf{R}^d)$ bounded from below.

Next, we use Kato's inequality (3.17) for $H_A^{(1)}$ to show that if $W(x) \geq 0$ is in $L^2_{\text{loc}}(\mathbf{R}^d)$, the range of $R + H_A^{(1)} + V + W$ is dense for $R > 0$ sufficiently large, i.e., that $H_A^{(1)} + V + W$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$.

Then we apply the arguments used by Faris-Lavine [FaLa-74]. The following lemma will be needed on the commutators of $H_A^{(1)}$ with the square roots of the exponential functions in (3.22) and (3.23) bounding $W(x)$ from below.

Lemma 3.7. *If $A(x)$ and $V(x)$ satisfy (3.22) (resp. (3.23)), then, for $\psi(x) = \exp[(a/2)(1+x^2)^{(1-b)/2}]$ with $a \geq 0$ and $0 < b \leq 1$ (resp. $\psi(x) = \exp[(a/2)(1+x^2)^{1/2}]$ with $0 \leq a/2 < m$), there exists a constant $C \geq 0$ such that*

$$\|[H_A^{(1)}, \psi]u\| \leq C\|\psi u\|, \quad u \in C_0^\infty(\mathbf{R}^d),$$

where $[H_A^{(1)}, \psi] := H_A^{(1)}\psi - \psi H_A^{(1)}$.

The proof of Lemma 3.7 is omitted and referred to [IIw-95].

We continue the proof of Theorem 3.6. Choose a nonnegative constant K such that $K \geq 2C$ and $V(x) + K\psi(x)^2 \geq 0$ a.e., where C and ψ are the same constant and the same function as in Lemma 3.7. Let $N = H^{(1)} + 3K\psi^2 = H_A^{(1)} + V + W + 3K\psi^2$, which is, as has been seen above, essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$. The closure of N is denoted by the same N . Then N satisfies $N \geq 2K\psi^2$ and has $C_0^\infty(\mathbf{R}^d)$ as its operator core. To prove essential selfadjointness of $H^{(1)}$ we have only to show that $\|H^{(1)}u\| \leq \|Nu\|$ for $u \in C_0^\infty(\mathbf{R}^d)$ and that $\pm i[H, N] \leq 3CN$ as the quadratic forms. This can be shown with the aid of the above lemma.

Finally we end the proof with the following note. When $A(x)$ and $V(x)$ satisfy (3.23), the above lemma appears to restrict the lower bound function $Ce^{a|x|}$ by $0 \leq a/2 < m$, but it is not so. To see this, for the moment only here we write $H_A^{(1)}$ in (3.7) as $H_{A,m}^{(1)}$ so as to manifest its m -dependence. Then recall [I4-92, Theorem 2.3] that the difference $H_{A,m}^{(1)} - H_{A,m'}^{(1)}$ is a bounded operator for all $m, m' \geq 0$. Therefore, if it is shown that $H_{A,m}^{(1)} + V + W$ is essentially selfadjoint for some $m \geq 0$, then it follows by the Kato–Rellich theorem that so is $H_{A,m'}^{(1)} + V + W$ for every $m' \geq 0$. \square

Results similar to Theorem 3.6 will hold for $H^{(2)}$ and $H^{(3)}$.

4 Imaginary-time path integrals for magnetic relativistic Schrödinger operators

It is well-known that the solution $u(x, t)$ of the Cauchy problem for the heat equation $\frac{d}{dt}u(x, t) = \left[\frac{1}{2}\Delta - V(x)\right]u(x, t)$ with initial data $u(x, 0) = g(x)$ can be represented by path integral, called *Feynman–Kac formula* (e.g. [RS-75, Theorem X.68, p.279], [Demuth–van Casteren [DvC-00], Theorem 2.5, p.61]:

$$u(x, t) = (e^{-t[-\frac{1}{2}\Delta + V]}g)(x) = \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-\int_0^t V(B(s))ds} g(B(t)) d\mu_x(B), \quad (4.1)$$

where, for each $x \in \mathbf{R}^d$, μ_x is the *Wiener measure* on the space $C_x([0, \infty) \rightarrow \mathbf{R}^d)$ of the Brownian paths which are continuous functions $B : [0, \infty) \rightarrow \mathbf{R}^d$ satisfying $B(0) = x$ (See also (4.29) below). The stochastic process concerned is called *Wiener process*. As $-\frac{1}{2}\Delta + V$ is a nonrelativistic Schrödinger operator with scalar potential $V(x)$ with mass 1, so the heat equation can be thought to be *imaginary-time Schrödinger equation*, because it is the equation to be obtained by starting from (real-time) Schrödinger equation $i\frac{\partial}{\partial t}\psi(x, t) = \left[-\frac{1}{2}\Delta + V(x)\right]\psi(x, t)$, next rotating it by -90° from real time t to imaginary time $-it$ in complex t -plane (see Figure 1) (cf. [I5-93, Section 4, p.23]) and then by *formally* putting $u(x, t) := \psi(x, -it)$, however, without seriously thinking about its meaning.

For a general nonrelativistic Schrödinger operator with vector and scalar potentials $A(x)$ and $V(x)$, $H^{NR} := H_A^{NR} + V := \frac{1}{2}(-i\nabla - A(x))^2 + V(x)$, there is also a path integral representation, called *Feynman–Kac–Itô formula* (e.g. Simon [S2-79/05]), for the solution $u(x, t) = (e^{-tH^{NR}}g)(x)$ of the Cauchy problem for the imaginary-time nonrelativistic Schrödinger equation, i.e. corresponding heat equation $\frac{d}{dt}u(x, t) =$

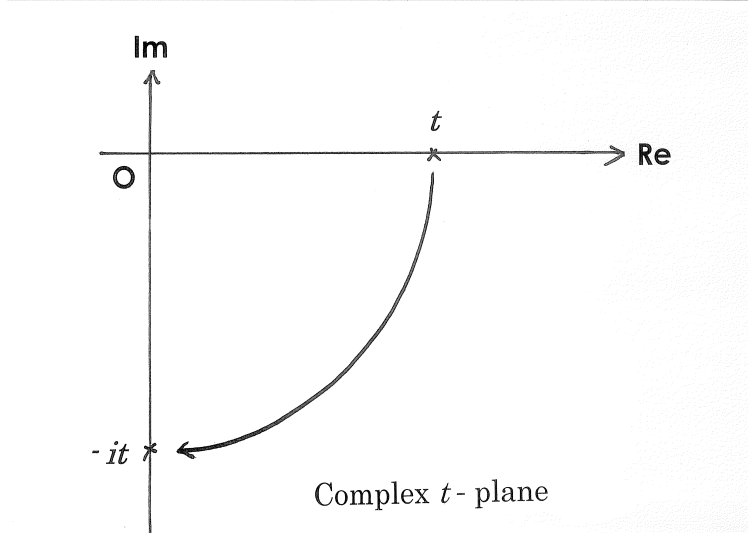


Figure 1: From real time t to imaginary time $-it$

$-H^{NR}u(x, t)$ with initial data $u(x, 0) = g(x)$:

$$\begin{aligned}
& (e^{-tH^{NR}}g)(x) \\
&= \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-[i \int_0^t A(B(s))dB(s) + \frac{i}{2} \int_0^t \text{div} A(B(s))ds + \int_0^t V(B(s))ds]} g(B(t)) d\mu_x(B) \\
&\equiv \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-[i \int_0^t A(B(s)) \circ dB(s) + \int_0^t V(B(s))ds]} g(B(t)) d\mu_x(B), \tag{4.2}
\end{aligned}$$

where $\int_0^t A(B(s))dB(s)$ is *Itô's integral* and $\int_0^t A(B(s)) \circ dB(s)$ the *Stratonovich integral*. In other words, we can say that these formulas (4.1) and (4.2) are representing the nonrelativistic Schrödinger semigroups $e^{-t[-\frac{1}{2}\Delta + V]}$ and $e^{-tH^{NR}}$.

In this section, we consider the same problem for the three magnetic relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$. In Section 4.1 we give path integral representations for their respective semigroups, and in Section 4.2 we discuss how these formulas are able to be deduced through some heuristic consideration.

4.1 Feynman–Kac–Itô type formulas for magnetic relativistic Schrödinger operators

Let H be one of the magnetic relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ in Definitions 2.1, 2.2, 2.3 or Definitions 3.2, 3.3, 3.4. In the same way as in the nonrelativistic case, rotating (real-time) relativistic Schrödinger equation $i \frac{\partial}{\partial t} \psi(x, t) = [H - m]\psi(x, t)$ by -90° from real time t to imaginary time $-it$ in complex t -plane (cf. [15, Section 4, p.23]), we arrive at the *imaginary-time relativistic Schrödinger equation*, i.e. the corresponding “heat equation” for $H - m$ [formally putting $u(x, t) := \psi(x, -it)$]:

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) &= -[H - m]u(x, t), & t > 0, \\ u(x, 0) &= g(x), & x \in \mathbf{R}^d. \end{cases} \tag{4.3}$$

The semigroup $u(x, t) = (e^{-t[H-m]}g)(x)$ gives the solution of this Cauchy problem as well. We want to deal with path integral representation for each $e^{-t[H^{(j)}-m]}g$ ($j = 1, 2, 3$). The relevant path integral is connected with the *Lévy process* (e.g. Ikeda–Watanabe [IkW2-81/89], Sato [Sa2-99], Applebaum [Ap-04/09]) on the space $D_x := D_x([0, \infty) \rightarrow \mathbf{R}^d)$, with each $x \in \mathbf{R}^d$, of the “*càdlàg* paths”, i.e. right-continuous paths $X : [0, \infty) \ni s \mapsto \mathbf{R}^d$ having left-hand limits and with $X(0) = x$. The associated path space measure is a probability measure λ_x , for each $x \in \mathbf{R}^d$, on $D_x([0, \infty) \rightarrow \mathbf{R}^d)$ whose characteristic function is given by

$$e^{-t[\sqrt{\xi^2+m^2}-m]} = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{i(X(t)-x) \cdot \xi} d\lambda_x(X), \quad t \geq 0, \quad \xi \in \mathbf{R}^d. \quad (4.4)$$

This path integral formula with measure λ_x was effectively used by [CaMS-90] to get asymptotic behavior of eigenfunctions for relativistic Schrödinger operator without vector potential. It also, together with the Feynman–Kac formula (4.2) with Wiener measure μ_x , was powerfully used in [ITak1-97, 2-98] to estimate in norm the difference between the Kac transfer operator $e^{-tV/2}e^{-tH_A}e^{-tV/2}$ and the nonrelativistic and/or relativistic Schrödinger semigroup $e^{-t(H_A+V)}$ by a power of t , in the case that H_A is a nonrelativistic magnetic Schrödinger operator H_A^{NR} and/ or a free relativistic Schrödinger operator $H_0 = \sqrt{-\Delta + 1} - 1$ with mass m .

We are going to start on task of representing the semigroup $e^{-t[H-m]}g$ by path integral. Before that, let us note that when the vector potential $A(x)$ is absent, we can represent $u(x, t)$ by a formula looking similar to the Feynman–Kac formula (4.1) for the nonrelativistic Schrödinger equation:

$$u(x, t) = (e^{-t[\sqrt{-\Delta+m^2}+V-m]}g)(x) = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-\int_0^t V(X(s))ds} g(X(t)) d\lambda_x(X). \quad (4.5)$$

Now, when the vector potential $A(x)$ is present, let us treat each case for $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$, separately.

(1) First consider the case for $H^{(1)} := H_A^{(1)} + V$ in Definition 3.3 with condition (3.9) on $A(x)$ and $V(x)$.

To represent $e^{-t[H^{(1)}-m]}g$ by path integral, we need some further notations from Lévy process.

For each path X , $N_X(dsdy)$ denotes the *counting measure* on $[0, \infty) \times (\mathbf{R}^d \setminus \{0\})$ to count the number of discontinuities of $X(\cdot)$, i.e.

$$N_X((t, t'] \times U) := \#\{s \in (t, t']; 0 \neq X(s) - X(s-) \in U\} \quad (4.6)$$

with $0 < t < t'$ and $U \subset \mathbf{R}^d \setminus \{0\}$ being a Borel set. It satisfies $\int_{D_x} N_X(dsdy) d\lambda_x(X) = ds n(dy)$. Put

$$\tilde{N}_X(dsdy) = N_X(dsdy) - ds n(dy), \quad (4.7)$$

which may be thought of as a renormalization of $N_X(dsdy)$. Then any path $X \in D_x([0, \infty) \rightarrow \mathbf{R}^d)$ can be expressed with $N_x(\cdot)$ and $\tilde{N}_X(\cdot)$ as

$$\begin{aligned} X(t) - x &= \int_0^{t+} \int_{|y| \geq 1} y N_X(dsdy) + \int_0^{t+} \int_{0 < |y| < 1} y \tilde{N}_X(dsdy) \\ &= \int_0^{t+} \int_{|y| > 0} y \tilde{N}_X(dsdy). \end{aligned} \quad (4.8)$$

Then we have the following path integral representation for $e^{-t[H^{(1)}-m]}g$.

Theorem 4.1. ([ITa-86], [I7-95]) *Assume that $A(x)$ and $V(x)$ satisfy condition (3.9). Then*

$$\begin{aligned}
(e^{-t[H^{(1)}-m]}g)(x) &= \int_{D_x([0,\infty)\rightarrow\mathbf{R}^d)} e^{-S^{(1)}(X;x,t)} g(X(t)) d\lambda_x(X), \\
S^{(1)}(X;x,t) &= i \int_0^{t+} \int_{|y|\geq 1} A(X(s-) + \frac{y}{2}) \cdot y N_X(dsdy) \\
&\quad + i \int_0^{t+} \int_{0<|y|<1} A(X(s-) + \frac{y}{2}) \cdot y \tilde{N}_X(dsdy) \\
&\quad + i \int_0^t \int_{0<|y|<1} [A(X(s) + \frac{y}{2}) - A(X(s))] \cdot y ds n(dy) + \int_0^t V(X(s)) ds \\
&= i \int_0^{t+} \int_{|y|>0} A(X(s-) + \frac{y}{2}) \cdot y \tilde{N}_X(dsdy) \\
&\quad + i \int_0^t \int_{|y|>0} [A(X(s) + \frac{y}{2}) - A(X(s))] \cdot y ds n(dy) + \int_0^t V(X(s)) ds.
\end{aligned} \tag{4.9}$$

Here it is easy to see the second equality in the expression (4.10) for $S^{(1)}(X;x,t)$, as well as in (4.8). We note also that, in (4.10), the integral in the third term of its second member is also written as the principal value integral:

$$\int_0^t \int_{0<|y|<1} [A(X(s) + \frac{y}{2}) - A(X(s))] \cdot y ds n(dy) = \int_0^t ds \text{ p.v. } \int_{0<|y|<1} A(X(s) + \frac{y}{2}) \cdot y n(dy),$$

and the same is valid for the second term of its third (last) member.

Proof of Theorem 4.1. We shall show first the case that both A and V are bounded and smooth, precisely, $A \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$ and $V \in C_b^\infty(\mathbf{R}^d; \mathbf{R})$, where, for l an positive integer, $C_b^\infty(\mathbf{R}^d; \mathbf{R}^l)$ is the Fréchet space of the \mathbf{R}^l -valued C^∞ functions in \mathbf{R}^d which together with their derivatives of all orders are bounded. Then we shall show the general case where they satisfy condition (3.9). Our proof follows the spirit of the proof of [RS-75, Theorem X.68, p.279] and [S2-79/05]. Representing the set Ω of “random variables ω ” by the path space $D_x([0, \infty) \rightarrow \mathbf{R}^d)$, we are suppressing use of “random variable ω ” by identifying it with path X .

I. The case that $A \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$ and $V \in C_b^\infty(\mathbf{R}^d; \mathbf{R})$.

Introduce a bounded operator $T(t)$ on $L^2(\mathbf{R}^d)$ by

$$\begin{aligned}
(T(t)g)(x) &:= \int_{\mathbf{R}^d} k_0(x-y, t) e^{iA(\frac{x+y}{2}) \cdot (x-y) - V(\frac{x+y}{2})t} g(y) dy \\
&= \int_{\mathbf{R}^d} k_0(x-y, t) e^{-iA(\frac{x+y}{2}) \cdot (y-x) - V(\frac{x+y}{2})t} g(y) dy
\end{aligned} \tag{4.11}$$

where $k_0(x-y, t)$ is the integral kernel of $e^{-t[\sqrt{-\Delta+m^2}-m]}$ in (3.3), which is nonnegative and satisfies $k_0(-x, t) = k_0(x, t)$. Then we can rewrite $T(t)$ as

$$(T(t)g)(x) = \int_{D_x} e^{-iA(\frac{x+X(t)}{2}) \cdot (X(t)-x) - V(\frac{x+X(t)}{2})t} g(X(t)) d\lambda_x(X). \tag{4.12}$$

Do partition of $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$, $t_j - t_{j-1} = t/n$, and put

$$S_n(x_0, \dots, x_n) := i \sum_{j=1}^n A\left(\frac{x_{j-1} + x_j}{2}\right) \cdot (x_{j-1} - x_j) + \sum_{j=1}^n V\left(\frac{x_{j-1} + x_j}{2}\right) \frac{t}{n}, \quad (4.13)$$

where $x_j = X(t_j)$ ($j = 0, 1, 2, \dots, n$); $x = x_0 = X(t_0) \equiv X(0)$, $y = x_n = X(t_n) \equiv X(t)$. Substitute these $n + 1$ points $x_j = X(t_j)$ on the path $X(\cdot)$ into $S_n(x_0, \dots, x_n)$ to get

$$\begin{aligned} S_n(X) &:= S_n(X(t_0), \dots, X(t_n)) \\ &= i \sum_{j=1}^n A\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \cdot (X(t_{j-1}) - X(t_j)) + \sum_{j=1}^n V\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \frac{t}{n}. \end{aligned} \quad (4.14)$$

Then we have

$$\begin{aligned} (T(t/n)^n g)(x) &= \overbrace{\int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d}}^{n \text{ times}} \prod_{j=1}^n k_0(x_{j-1} - x_j, t/n) e^{-S_n(x_0, \dots, x_n)} g(x_n) dx_1 \dots dx_n \\ &= \int_{D_x} e^{-S_n(X)} g(X(t)) d\lambda_x(X), \quad x_0 = x. \end{aligned} \quad (4.15)$$

Before we continue further the proof of Theorem 4.1, we show the following proposition which refers to the convergence of the left-hand side of (4.15).

Proposition 4.2. $T(t/n)^n$ converges strongly to $e^{-t[H^{(1)} - m]}$ in $L^2(\mathbf{R}^d)$ as $n \rightarrow \infty$.

Proof. Since the operators $T(t/n)^n$ are uniformly bounded, we have only to show that $T(t/n)^n g$ is convergent to the limit in $L^2(\mathbf{R}^d)$ for g in the domain $D[H^{(1)}] = H^1(\mathbf{R}^d)$ of $H^{(1)}$. We have with $M = \sup_{x \in \mathbf{R}^d} |V(x)|$

$$\begin{aligned} &\|T(t/n)^n g - e^{-t[H^{(1)} - m]} g\| \\ &= \left\| \sum_{j=1}^n T(t/n)^{j-1} (T(t/n) - e^{-(t/n)[H^{(1)} - m]}) e^{-(n-j)(t/n)[H^{(1)} - m]} g \right\| \\ &\leq e^{Mt} \sup_{0 \leq s \leq t} n \| (T(t/n) - e^{-(t/n)[H^{(1)} - m]}) e^{-s[H^{(1)} - m]} g \|, \end{aligned}$$

which we can show tends to zero uniformly on each bounded t -interval in $[0, \infty)$ as $n \rightarrow \infty$. To see this, we show first that $(d/d\tau)(T(\tau)g)$ converges to $-[H - m]g$ in L^2 , as $\tau \downarrow 0$. Indeed, we have by (3.4)

$$\begin{aligned} &\int ([\sqrt{-\Delta_x + m^2} - m] k_0(x - y, \tau)) e^{-iA\left(\frac{x+y}{2}\right) \cdot (y-x) - V\left(\frac{x+y}{2}\right) \tau} g(y) dy \\ &= - \int \int_{|z| > 0} [k_0(x + z - y, \tau) - k_0(x - y, \tau) - I_{\{|z| < 1\}} z \cdot \nabla_x k_0(x - y, \tau)] n(dz) \\ &\quad \times e^{-iA\left(\frac{x+y}{2}\right) \cdot (y-x) - V\left(\frac{x+y}{2}\right) \tau} g(y) dy \\ &= - \int \int_{|z| > 0} \left[k_0(x - y, \tau) e^{-iA\left(\frac{x+z+y}{2}\right) \cdot (z+y-x) - V\left(\frac{x+z+y}{2}\right) \tau} g(z + y) \right. \\ &\quad \left. - \left(k_0(x - y, \tau) - I_{\{|z| < 1\}} z \cdot \nabla_x k_0(x - y, \tau) \right) e^{-iA\left(\frac{x+y}{2}\right) \cdot (y-x) - V\left(\frac{x+y}{2}\right) \tau} g(y) \right] n(dz) dy, \end{aligned}$$

where we have changed the variable $z - y =: -y'$ and then rewritten y for y' again. Then noting that $\nabla_x k_0(x - y, \tau) = -\nabla_y k_0(x - y, \tau)$ and integrating by parts in the variable y , we see that the above integral converges to

$$- \int_{|z|>0} [e^{-iz \cdot A(x + \frac{z}{2})} g(x + z) - g(x) - I_{\{|z|<1\}} z \cdot \nabla_x g(x)] n(dz),$$

as $\tau \rightarrow +0$, because then $k_0(x - y, \tau) \rightarrow \delta(x - y)$. It follows that, as $\tau \rightarrow +0$,

$$\begin{aligned} & \left(\frac{\partial}{\partial \tau} T(\tau) g \right)(x) \\ &= - \int_{|y|>0} \left[\left([\sqrt{-\Delta_x + m^2} - m] + V\left(\frac{x+y}{2}\right) k_0(x - y, \tau) \right) e^{-iA\left(\frac{x+y}{2}\right) \cdot (y-x) - V\left(\frac{x+y}{2}\right) \tau} g(y) dy \right] \end{aligned}$$

converges to

$$- \int_{|z|>0} [e^{-iA(x + \frac{z}{2}) \cdot z} g(x + z) - g(x) - I_{\{|z|<1\}} z \cdot \nabla_x g(x)] n(dz) - V(x) g(x) = (-[H^{(1)} - m] g)(x).$$

Thus we see that for $h \in D[H^{(1)}]$

$$\begin{aligned} n \| [T(t/n) - e^{-(t/n)[H^{(1)} - m]}] h \| &= n \left\| \int_0^{t/n} \frac{\partial}{\partial \tau} [T(\tau) - e^{-\tau[H^{(1)} - m]}] h d\tau \right\| \\ &= n \left\| \int_0^{t/n} \left(\frac{\partial}{\partial \tau} T(\tau) + [H^{(1)} - m] e^{-\tau[H^{(1)} - m]} \right) h d\tau \right\| \\ &\leq t \sup_{0 \leq \tau \leq t/n} \left\| \left(\frac{\partial}{\partial \tau} T(\tau) + [H^{(1)} - m] e^{-\tau[H^{(1)} - m]} \right) h \right\| \end{aligned}$$

converges to zero as $n \rightarrow \infty$. Moreover, this convergence is uniform on compact subsets in $t \geq 0$ as $n \rightarrow \infty$. Noting $D[H^{(1)}]$ is a Hilbert space $D[H^{(1)}]$ with graph norm of $H^{(1)} - m$, we see by uniform boundedness principle that the sequence $\{n(T(t/n) - e^{-(t/n)[H^{(1)} - m]}]\}_{n=1}^\infty$ is, as a family of bounded operators of the Hilbert space $D[H^{(1)}]$ into $L^2(\mathbf{R}^d)$, uniformly bounded for all n and on every fixed compact subset in $t \geq 0$. Consequently, it converges to zero uniformly on compact subsets of the Hilbert space $D[H^{(1)}]$. The map $[0, t] \ni s \mapsto e^{-s[H^{(1)} - m]} \in D[H^{(1)}]$ is continuous, so that $\{e^{-s[H^{(1)} - m]} g; 0 \leq s \leq t\}$ is a compact subset of $D[H^{(1)}]$. This shows Proposition 4.2. \square

We continue the proof of Theorem 4.1, I.

To see the convergence of the right-hand side of (4.15), put

$$S_n(X) = S_{n1}(X) + S_{n2}(X), \quad (4.16)$$

$$S_{n1}(X) = i \sum_{j=1}^n A\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \cdot (X(t_j) - X(t_{j-1})), \quad (4.17)$$

$$S_{n2}(X) = \sum_{j=1}^n V\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) (t_j - t_{j-1}). \quad (4.18)$$

First, for $S_{n2}(X)$ in (4.18), it is evident that for each $X \in D_x$, $S_{n2}(X)$ converges to $\int_0^t V(X(s)) ds$, i.e. the last term of the second member of (4.10), as $n \rightarrow \infty$.

Next, to see the convergence of $S_{n1}(X)$ in (4.17) to the sum of the other three terms in the same (second) member of (4.10) which involve $A(\cdot)$, we rewrite by Itô's formula [IkW2-81/89, Chap.II, 5, Theorem 5.1] (cf. (4.8)) the summand in $S_{n1}(X)$ as

$$\begin{aligned}
& A\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \cdot (X(t_j) - X(t_{j-1})) \\
&= \int_{t_{j-1}}^{t_j} \int_{|y|>0} \left[A\left(\frac{X(s-) + X(t_{j-1}) + yI_{|y|\geq 1}(y)}{2}\right) \cdot (X(s-) - X(t_{j-1}) + yI_{\{|y|\geq 1\}}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s-) + X(t_{j-1})}{2}\right) \cdot (X(s-) - X(t_{j-1})) \right] N_X(dsdy) \\
&+ \int_{t_{j-1}}^{t_j} \int_{|y|>0} \left[A\left(\frac{X(s-) + X(t_{j-1}) + yI_{|y|<1}(y)}{2}\right) \cdot (X(s-) - X(t_{j-1}) + yI_{\{|y|<1\}}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s-) + X(t_{j-1})}{2}\right) \cdot (X(s-) - X(t_{j-1})) \right] \tilde{N}_X(dsdy) \\
&+ \int_{t_{j-1}}^{t_j} \int_{|y|>0} \left\{ A\left(\frac{X(s) + X(t_{j-1}) + yI_{|y|<1}(y)}{2}\right) \cdot (X(s) - X(t_{j-1}) + yI_{\{|y|<1\}}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s) + X(t_{j-1})}{2}\right) \cdot (X(s) - X(t_{j-1})) \right. \\
&\quad \left. - I_{\{|y|<1\}}(y) \left[\left(\frac{1}{2}(y \cdot \nabla)A\right)\left(\frac{X(s) + X(t_{j-1})}{2}\right) \cdot (X(s) - X(t_{j-1})) \right. \right. \\
&\quad \left. \left. + y \cdot A\left(\frac{X(s) + X(t_{j-1})}{2}\right) \right] \right\} ds n(dy).
\end{aligned}$$

Then, taking $n = 2^k$ so that $t_j = 2^{-k}jt$, $j = 0, 1, \dots, 2^k$, we can see for each $X \in D_x$ that as $k \rightarrow \infty$, $S_{n1}(X)$ converges to the sum of the first, second and third terms in the second member of (4.10). Thus $S_n(X)$ in (4.14)/(4.16) converges to the second member of (4.10), therefore $S^{(1)}(X; x, t)$. As a result, by the Lebesgue dominated convergence theorem the right-hand side of (4.15) converges to the right-hand side of (4.9).

II. The general case where $A(x)$ and $V(x)$ satisfy condition (3.9).

Choose a sequence $\{A_k\}$ in $C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$ with $|A_k(x)| \leq |A(x)|$ which is convergent to $A(x)$ in $L_{loc}^{1+\delta}$ and pointwise a.e., and a sequence $\{V_k\}$ in $C_b^\infty(\mathbf{R}^d; \mathbf{R})$ with $0 \leq V_k(x) \leq V(x)$ which is convergent to $V(x)$ in L_{loc}^1 and pointwise a.e., as $k \rightarrow \infty$. Then by (4.9), (4.10) we have

$$(e^{-t[H_k^{(1)} - m]}g)(x) = \int_{D_x} e^{-S_k(X; x, t)} g(X(t)) d\lambda_x(X), \quad (4.19)$$

where $S_k(X; x, t)$ (though here with superscript (1) removed, for notational simplicity) is the $S^{(1)}(X; x, t)$ in (4.10) with A_k and V_k in place of A and V , and $H_k^{(1)}$ is the selfadjoint operator associated with the form $h_k^{(1)} \equiv h_{A_k, V_k}^{(1)}$ in (2.5). We shall show both sides of (4.19) converge to those of (4.9) as $k \rightarrow \infty$.

As far as the left-hand side of (4.19) is concerned, by [ITs2-93, Lemma 3.6], $H_k^{(1)}$ converges to $H^{(1)}$ in the strong resolvent sense, and by [Kat-76, IX, Theorem 2.16, p.504], $\{\exp[-t(H_k^{(1)} - m)]g\}_{k=1}^\infty$ converges to $\exp[-t(H^{(1)} - m)]g$, uniformly on each bounded t -interval in $[0, \infty)$, in L^2 and, if a subsequence is taken, pointwise a.e.

To see convergence of the right-hand side of (4.19), we shall show that $\{\exp[-S_k(X; x, t)]\}_{k=1}^\infty$ converges for a.e. x and λ_x -a.e. X , as $k \rightarrow \infty$, and its limit can be written as $e^{-S^{(1)}(X; x, t)}$. Put

$$S_k(X; x, t) = \sum_{j=1}^4 S_k^{(j)}(X; x, t), \quad S^{(1)}(x, t) = \sum_{j=1}^4 S^{(j)}(X; x, t).$$

We show each $\exp[-S_k^{(j)}(X; x, t)]$ ($j = 1, 2, 3, 4$) converges for λ_x -a.e. X .

(i) There exists a Borel set K_1 in \mathbf{R}^d of Lebesgue measure zero such that $|A(x)|$ is finite and $A_k(x) \rightarrow A(x)$ for $x \notin K_1$. Then for each (s, y) with $0 < s \leq t$ and $|y| \geq 1$,

$$G_1(s, y) := \{X \in D_x; X(s-) + y/2 \in K_1\}$$

has λ_x -measure zero, because

$$\int_{G(s, y)} d\lambda_x(X) = \int_{K_1 - y/2} k_0(s, x - z) dz = 0.$$

Therefore by the Fubini theorem

$$G_1 := \{(X, s, y) \in D_x \times (0, t] \times \{|y| \geq 1\}; X(s-) + y/2 \in K_1\}$$

has $[d\lambda_x \times ds n(dy)]$ -measure zero, because

$$\int \int \int I_{G_1}(X, s, y) d\lambda_x ds n(dy) = \int_0^t \int_{|y| \geq 1} ds n(dy) \int_{G_1(s, y)} d\lambda_x(X) = 0,$$

where $I_{G_1}(X, s, y)$ is the indicator function for the set G_1 . It follows again by the Fubini theorem that for λ_x -a.e. X ,

$$G_1(X) := \{(s, y) \in (0, t] \times \{|y| \geq 1\}; X(s-) + y/2 \in K_1\}$$

has $N_X(ds dy)$ -measure zero, since

$$\int_{D_x} d\lambda_x(X) \int \int_{G_1(X)} N_X(ds dy) = \int_{D_x} d\lambda_x(X) \int \int_{G_1(X)} ds n(dy).$$

Therefore for λ_x -a.e. X , as $k \rightarrow \infty$,

$$A_k(X(s-) + y/2) \rightarrow A(X(s-) + y/2), \quad N_X(ds dy) - \text{a.e.},$$

and the integral $S_k^{(1)}(X; x, t)$ exists, being a finite sum because $X(s)$ has at most finitely many discontinuities s with the jump $|X(s) - X(s-)|$ exceeding a given positive constant. By the Lebesgue dominated convergence theorem, for λ_x -a.e. X , $S_k^{(1)}(X; x, t) \rightarrow S^{(1)}(X; x, t)$ and hence $\exp[-S_k^{(1)}(X; x, t)] \rightarrow \exp[-S^{(1)}(X; x, t)]$.

(ii) For $n > 0$ let $\sigma_n(X) = \inf\{s > 0; |X(s-)| > n\}$. Then for λ_x -a.e. X , $\lim_{n \rightarrow \infty} \sigma_n(X) = \infty$. For k, l integers put $A_{kl}(x) = A_k(x) - A_l(x)$, and

$$G_2^{kl} := \{(X, s, y) \in D_x \times (0, t] \times \{0 < |y| < 1\}; |A_{kl}(X(s-) + y/2) \cdot y| > 1\}$$

and for each $X \in D_x$

$$G_2^{kl}(X) := \{(s, y) \in (0, t] \times \{0 < |y| < 1\}; |A_{kl}(X(s-) + y/2) \cdot y| > 1\}.$$

The complements of G_2^{kl} in the set $D_x \times (0, t] \times \{0 < |y| < 1\}$ and $G_2^{kl}(X)$ in the set $(0, t] \times \{0 < |y| < 1\}$ are denoted by $(G_2^{kl})^c$ and $(G_2^{kl}(X))^c$, respectively. Then we have, for n fixed and for an arbitrary compact subset K of \mathbf{R}^d with Lebesgue measure $|K|$,

$$\begin{aligned} & \int_K dx \int_{D_x} |S_k^{(2)}(X; x, t \wedge \sigma_n(X)) - S_l^{(2)}(X; x, t \wedge \sigma_n(X))| d\lambda_x(X) \\ & \leq \int_K dx \int_{D_x} \left| \int \int_{G_2^{kl}(X)} I_{[0, \sigma_n(X)]}(s) A_{kl}(X(s-) + y/2) \cdot y \tilde{N}_X(ds dy) \right| d\lambda_x(X) \\ & \quad + \int_K dx \int_{D_x} \left| \int \int_{(G_2^{kl}(X))^c} I_{[0, \sigma_n(X)]}(s) A_{kl}(X(s-) + y/2) \cdot y \tilde{N}_X(ds dy) \right| d\lambda_x(X) \\ & \equiv \int_K I_1^{kl} dx + \int_K I_2^{kl} dx. \end{aligned}$$

For I_1^{kl} we have

$$\begin{aligned} \int_K I_1^{kl} dx & \leq \int_K dx \int_{D_x} d\lambda_x(X) \\ & \quad \times \int \int_{G_2^{kl}(X)} I_{[0, \sigma_n(X)]}(s) |A_{kl}(X(s-) + y/2) \cdot y| (N_X(ds dy) + ds n(dy)) \\ & \leq 2 \int_K dx \int_{D_x} d\lambda_x(X) \int \int_{G_2^{kl}(X)} I_{[0, \sigma_n(X)]}(s) |A_{kl}(X(s) + y/2) \cdot y|^{1+\delta} ds n(dy) \\ & \leq 2 \int_K dx \int_0^t \int_{0 < |y| < 1} ds n(dy) \int_{|z| \leq n} |A_{kl}(z + y/2) \cdot y|^{1+\delta} k_0(s, x - z) dz \\ & \leq 2n_\delta t \int_{|z| \leq n+1} |A_{kl}(z)|^{1+\delta} dz, \end{aligned}$$

with $n_\delta = \int_{0 < |y| < 1} |y|^{1+\delta} n(dy)$, where in the second inequality we have used that $|A_{kl}(X(s) + y/2) \cdot y| > 1$ on $G_2^{kl}(X)$.

For I_2^{kl} we have

$$\begin{aligned} (I_2^{kl})^2 & = \int_{D_x} d\lambda_x(X) \int \int_{(G_2^{kl}(X))^c} I_{[0, \sigma_n(X)]}(s) |A_{kl}(X(s) + y/2) \cdot y|^2 ds n(dy) \\ & \leq \int_{D_x} d\lambda_x(X) \int \int_{(G_2^{kl}(X))^c} I_{[0, \sigma_n(X)]}(s) |A_{kl}(X(s) + y/2) \cdot y|^{1+\delta} ds n(dy) \\ & = \int_0^t \int_{0 < |y| < 1} ds n(dy) \int_{|z| \leq n} |A_{kl}(z + y/2) \cdot y|^{1+\delta} k_0(s, x - z) dz, \end{aligned}$$

where the inequality is due to that $|A_{kl}(X(s) + y/2) \cdot y| \leq 1$ on $(G_2^{kl}(X))^c$. Hence

$$\begin{aligned} \int_K I_2^{kl} dx &\leq |K|^{1/2} \left(\int (I_2^{kl})^2 dx \right)^{1/2} \\ &= |K|^{1/2} \left(\int dx \int_0^t \int_{0 < |y| < 1} ds n(dy) \right. \\ &\quad \left. \times \int_{|z| \leq n} |A_{kl}(z + y/2) \cdot y|^{1+\delta} k_0(s, x - z) dz \right)^{1/2} \\ &\leq (|K| n_\delta t)^{1/2} \left(\int_{|z| \leq n+1} |A_{kl}(z)|^{1+\delta} dz \right)^{1/2}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_K dx \int_{D_x} |S_k^{(2)}(X; x, t \wedge \sigma_n(X)) - S_l^{(2)}(X; x, t \wedge \sigma_n(X))| d\lambda_x(X) \\ &\leq \int_K I_1^{kl} dx + \int_K I_2^{kl} dx \\ &\leq 2n_\delta t \int_{|z| \leq n+1} |A_{kl}(z)|^{1+\delta} dz + (|K| n_\delta t)^{1/2} \left(\int_{|z| \leq n+1} |A_{kl}(z)|^{1+\delta} dz \right)^{1/2}, \end{aligned}$$

which tends to zero as $k, l \rightarrow \infty$. Since K is arbitrary, it can be seen, by passing to a subsequence, for a.e. x , that as $k \rightarrow \infty$, $\{S_k^{(2)}(X; x, t \wedge \sigma_n(X))\}_{k=1}^\infty$ converges to a limit in L^1 with respect to λ_x and, by passing to a subsequence, for λ_x -a.e. X . This limit is what is to be denoted by $S^{(2)}(X; x, t \wedge \sigma_n(X))$. Further, since $\lim_{n \rightarrow \infty} \sigma_n(X) = \infty$ for λ_x -a.e. X , we see that $\{S_k^{(2)}(X; x, t)\}_{k=1}^\infty$ converges to a limit for λ_x -a.e. X , which is to be denoted by $S^{(2)}(X; x, t)$, and hence $\exp[-S_k^{(2)}(X; x, t)] \rightarrow \exp[-S^{(2)}(X; x, t)]$.

(iii) We can show with the theory of singular integrals that

$$a(x) := \text{p.v.} \int_{0 < |y| < 1} A(x + y/2) \cdot y n(dy)$$

exists pointwise a.e. in x , while

$$a_k(x) := \int_{0 < |y| < 1} (A_k(x + y/2) - A_k(x)) \cdot y n(dy) = \text{p.v.} \int_{0 < |y| < 1} A_k(x + y/2) \cdot y n(dy)$$

exists for every x , and that as $k \rightarrow \infty$, $a_k(x)$ converges to $a(x)$ in $L_{loc}^{1+\delta}$. With the same $\sigma_n(X)$ as in (ii), we have, for n fixed,

$$\begin{aligned} &\int_{\mathbf{R}^d} dx \int_{D_x} |S_k^{(3)}(X; x, t \wedge \sigma_n(X)) - S_l^{(3)}(X; x, t \wedge \sigma_n(X))|^{1+\delta} d\lambda_x(X) \\ &= \int dx \int d\lambda_x(X) \left| \int_0^t I_{[0, \sigma_n(X)]}(s) [a_k(X(s)) - a_l(X(s))] ds \right|^{1+\delta} \\ &\leq \int dx \int d\lambda_x(X) \left(\int_0^t ds \right)^\delta \int_0^t I_{[0, \sigma_n(X)]}(s) |a_k(X(s)) - a_l(X(s))|^{1+\delta} ds \\ &\leq t^\delta \int dx \int_0^t ds \int_{|z| \leq n} |(a_k(z) - a_l(z))|^{1+\delta} k_0(s, x - z) dz \\ &\leq t^{1+\delta} \int_{|x| \leq n} |(a_k(x) - a_l(x))|^{1+\delta} dx \rightarrow 0, \quad k, l \rightarrow \infty, \end{aligned}$$

where in the first inequality we have used the Hölder inequality. Similarly to (ii), it can be seen, by passing to a subsequence, for a.e. x , that as $k \rightarrow \infty$, $\{S_k^{(3)}(X; x, t \wedge \sigma_n(X))\}_{k=1}^\infty$ converges to a limit in $L^{1+\delta}$ with respect to λ_x and, by passing to a subsequence, for λ_x -a.e. X . This limit is what is to be denoted by $S^{(3)}(X; x, t \wedge \sigma_n(X))$. Further, since $\lim_{n \rightarrow \infty} \sigma_n(X) = \infty$ for λ_x -a.e. X , we see that $\{S_k^{(3)}(X; x, t)\}_{k=1}^\infty$ converges to a limit for λ_x -a.e. X , which is to be denoted by $S^{(3)}(X; x, t)$, and hence $\exp[-S_k^{(3)}(X; x, t)] \rightarrow \exp[-S^{(3)}(X; x, t)]$.

(iv) The proof will proceed in the same way as the proof of [S2-79/05, II, Theorem 6.2, p.51]. We may suppose that $V_k(x) \uparrow V(x)$ pointwise a.e. There exists a Borel set K_4 in \mathbf{R}^d of Lebesgue measure zero such that $V_n(x) \uparrow V(x)$ for $x \notin K_4$. Then for $0 < s \leq t$,

$$G_4(s, y) = \{X \in D_x; X(s) \in K_4\}$$

has λ_x -measure zero. Therefore by the Fubini theorem

$$G_4 = \{(X, s) \in D_x \times (0, t]; X(s) \in K_4\}$$

has $[d\lambda_x \times ds]$ -measure zero, so that for λ_x -a.e. X ,

$$G_4(X) = \{s \in (0, t]; X(s) \in K_4\}$$

has Lebesgue measure zero. It follows by the monotone convergence theorem that for λ_x -a.e. X , $S_n^{(4)}(X; x, t) \rightarrow S^{(4)}(X; x, t)$ and hence $\exp[-S_n^{(4)}(X; x, t)] \rightarrow \exp[-S^{(4)}(X; x, t)]$. This proves Theorem 4.1. \square

(2) Next we come to the case for $H^{(2)} := H_A^{(2)} + V$ in Definition 3.3 with condition (3.9) on $A(x)$ and $V(x)$.

Theorem 4.3. [IfMP1-07, 2-08, 3-10] *The same hypothesis as in Theorem 4.1.*

$$(e^{-t[H^{(2)}-m]}g)(x) = \int_{D_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{-S^{(2)}(X;x,t)} g(X(t)) d\lambda_x(X), \quad (4.20)$$

$$\begin{aligned} S^{(2)}(X; x, t) &= i \int_0^{t+} \int_{|y|>0} \left(\int_0^1 A(X(s-) + \theta y) d\theta \right) \cdot y \tilde{N}_X(ds dy) \\ &\quad + i \int_0^t \int_{|y|>0} \left[\int_0^1 A(X(s) + \theta y) d\theta - A(X(s)) \right] \cdot y ds n(dy) + \int_0^t V(X(s)) ds. \end{aligned} \quad (4.21)$$

The proof of Theorem 4.3 can be done in exactly the same way as that of Theorem 4.1. Indeed, we have only to replace $A(X(s-) + \frac{y}{2}) \cdot y$ by $(\int_0^1 A(X(s-) + \theta y) d\theta) \cdot y$. We will not repeat it here.

(3) Finally, we consider the case for the operator defined, in Definition 3.4, with the square root of a nonnegative selfadjoint operator, $H^{(3)} := H_A^{(3)} + V$.

On the one hand, we can determine by functional analysis, namely, by theory of fractional powers (e.g. Yosida [Y, Chap. IX, 11, pp.259–261]) $e^{-t[H_A^{(3)}-m]}$ from the nonnegative selfadjoint operator $S := (-i\nabla - A(x))^2 + m^2 =: 2H_A^{NR} + m^2$, where

$H_A^{NR} = \frac{1}{2}(-i\nabla - A(x))^2$ is the magnetic nonrelativistic Schrödinger operator with mass 1 and without scalar potential. Indeed, we have

$$\begin{aligned} e^{-t[H_A^{(3)}-m]}g &= \begin{cases} e^{mt} \int_0^\infty f_t(\kappa) e^{-\kappa S} g d\kappa, & t > 0, \\ 0, & t = 0 \end{cases} \\ f_t(\kappa) &= \begin{cases} (2\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\kappa-tz^{1/2}} dz, & \kappa \geq 0, \\ 0, & \kappa < 0 \quad (\sigma > 0). \end{cases} \end{aligned} \quad (4.22)$$

This equation (4.22) may provide a kind of path integral representation for $e^{-t[H_A^{(3)}-m]}g$ with the Wiener measure μ_x^1 corresponding to mass 1 [cf. $\mu_x^m \equiv \mu_x$ in (4.29)]:

$$\begin{aligned} &(e^{-t[H_A^{(3)}-m]}g)(x) \\ &= e^{mt} \int_0^\infty d\kappa f_t(\kappa) e^{-\kappa m^2} \\ &\quad \times \int_{C_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{-[i \int_0^{2\kappa} A(B(s)) \odot dB(s) + \int_0^{2\kappa} V(B(s)) ds]} g(B(2\kappa m)) d\mu_x^1(B), \end{aligned}$$

though with an undesirable extra $d\kappa$ -integral, by substituting the Feynman–Kac–Itô formula (4.2) with $V = 0$, i.e. for $e^{-t[H_A^{(3)}-m]}g$ with $t = 2\kappa$ into $e^{-\kappa(S-m^2)} = e^{-2\kappa H_A^{NR}}$ in the integrand of (4.22).

Then if we would use this further to represent $e^{-t[H^{(3)}-m]}g$ for $V \neq 0$, we might apply the Trotter–Kato product formula

$$e^{-t[H^{(3)}-m]} = \text{s-}\lim_{n \rightarrow \infty} \left(e^{-(t/n)[H_A^{(3)}-m]} e^{-(t/n)V} \right)^n, \quad (4.23)$$

for the sum $H^{(3)} - m = (H_A^{(3)} - m) + V$ to express the semigroup $e^{-t[H^{(3)}-m]}$ as a “limit”, where convergence of the right-hand side usually takes place in strong operator topology as indicated. However it is not clear whether this procedure could further yield a path integral representation for $e^{-t[H^{(3)}-m]}g$.

In passing, let us insert a comment on the convergence of Trotter–Kato product formula (4.23). It is now known that the convergence takes place even *in operator norm*, so long as the operator sum $(H_A^{(3)} - m) + V$ is selfadjoint on the common domain $D[H_A^{(3)}] \cap D[V]$ by the recent results in Ichinose–Tamura [IT1-01], Ichinose–Tamura–Tamura–Zagrebnov [ITTaZ-01] and also even pointwise convergence of the integral kernels in [IT2-04, 3-06] (cf. [I8-99]).

On the other hand, it does not seem possible to represent $e^{-t[H^{(3)}-m]}g$ by path integral through directly applying Lévy process as we saw in the cases for $e^{-t[H^{(1)}-m]}g$ and $e^{-t[H^{(2)}-m]}g$, because $H_A^{(3)}$ does not seem to be explicitly expressed by a pseudo-differential operator corresponding to a certain tractable symbol. It was in this situation that the problem of path integral representation for $e^{-t[H^{(3)}-m]}g$ was studied first by DeAngelis–Serva [DeSe-90] and DeAngelis–Rinaldi–Serva [DeRSe-91] with use of *subordination /time-change* of Brownian motion, and then by Nagasawa [N1-96, 2-97, 3-00]. Recently it has been more extensively studied by Hiroshima–Ichinose–Lórcinzi [HILo1-12, 2-12] (cf. [LoHB-11]) not only for the magnetic relativistic Schrödinger operator $H_A^{(3)}$ but also for Bernstein functions of the magnetic nonrelativistic Schrödinger operator and even with spin. In this connection, the problem on nonrelativistic limit was studied in [I1-87], [Sa1-90], [N1-97].

To proceed, let us explain about subordination (e.g. [Sa2, Chap.6, p.197], [Ap-04/09, 1.3.2, p.52]). Subordination is a transformation, through random time change, of a stochastic process to a new one which is a non-decreasing Lévy process independent of the original one, what is called *subordinator*. The new process is said to be *subordinate* to the original one.

As the original process, take $B^1(t)$, the one-dimensional standard Brownian motion, so that $B^1 \equiv B^1(\cdot)$ is a function belonging to the space $C_0([0, \infty) \rightarrow \mathbf{R})$ of real-valued continuous functions on $[0, \infty)$ satisfying $B^1(0) = 0$ and

$$e^{-t\frac{\xi^2}{2}} = \int_{C_0([0, \infty) \rightarrow \mathbf{R})} e^{i\xi B^1(t)} d\mu_0^S(B^1),$$

where μ_0^S is the Wiener measure on $C_0([0, \infty) \rightarrow \mathbf{R})$. Let $m \geq 0$, and for each B^1 and $t \geq 0$, put

$$T(t) \equiv T(t, B^1) := \inf\{s > 0; B^1(s) + ms = t\}. \quad (4.24)$$

Then $T \equiv T(\cdot)$ is a monotone, non-decreasing function on $[0, \infty)$ with $T(0) = 0$, belonging to $D_0([0, \infty) \rightarrow \mathbf{R})$ and so becoming a one-dimensional Lévy process, called *inverse Gaussian subordinator* for $m > 0$ and *Lévy subordinator* for $m = 0$. This correspondence defines a map \hat{T} of $C_0([0, \infty) \rightarrow \mathbf{R})$ into $D_0([0, \infty) \rightarrow \mathbf{R})$ by $\hat{T}B^1(\cdot) = T(\cdot, B^1)$. Let ν_0 be the probability measure on $D_0([0, \infty) \rightarrow \mathbf{R})$ defined by $\nu_0(G) = \mu_0^S(\hat{T}^{-1}G)$ first for cylinder subsets $G \subset D_0([0, \infty) \rightarrow \mathbf{R})$ and then extended to more general subsets.

Proposition 4.4. (e.g. [Ap-04/09, Example 1.3.21, p.54, and Exercise 2.2.10, p.96; cf. Theorem 2.2.9, p.95]) *The probability measure ν_0 satisfies*

$$e^{-t[\sqrt{2\sigma+m^2}-m]} = \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{-T(t)\sigma} d\nu_0(T), \quad \sigma \geq 0. \quad (4.25)$$

Proof. The proof will be not selfcontained, and need some basic facts about *martingale* and *stopping time* (e.g. [IkW2-81/89], [Sa2-99], [Ap-04/09], [DvC-00], [LoHB-11]). $T(t, B^1) = T(t)$ is a stopping time and then $B^1(T(t))$ is a *stopped random variable* belonging to $D_0([0, \infty) \rightarrow \mathbf{R})$. Then we see that, for $\theta \in \mathbf{R}$,

$$M_\theta(t) := e^{\theta B^1(t) - \frac{1}{2}\theta^2 t}$$

is a continuous martingale with respect to the natural filtration of Brownian motion $B^1(t)$. Further,

$$M_\theta((\hat{T}B^1)(t) \wedge n) = e^{B^1(T(t) \wedge n) - \frac{1}{2}\theta^2 T(t) \wedge n}$$

is also a martingale. Then by *Doob's optional stopping theorem* [Ap-04/09, Theorem 2.2.1, p.92], for each $t > 0$, positive integer n and $\theta \geq 0$, we have

$$\begin{aligned} \int_{C_0([0, \infty) \rightarrow \mathbf{R})} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) &= \int_{C_0([0, \infty) \rightarrow \mathbf{R})} M_\theta((\hat{T}B^1)(0) \wedge n) d\mu_0^S(B^1) \\ &= \int_{C_0([0, \infty) \rightarrow \mathbf{R})} e^{\theta B^1(0)} d\mu_0^S(B^1) = 1. \end{aligned}$$

For each positive integer n and $t > 0$, put $\Omega_{n,t} := \{B^1 \in C_0([0, \infty) \rightarrow \mathbf{R}); T(t) = (\hat{T}B^1)(t) \leq n\}$. Then

$$\begin{aligned} & \int_{C_0([0, \infty) \rightarrow \mathbf{R})} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) \\ &= \int_{\Omega_{n,t}} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) + \int_{\Omega_{n,t}^c} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1). \end{aligned}$$

Since, for $B^1(t)$, $(\hat{T}B^1)(t) = T(t) > n$ implies $B^1(n) < t - mn$, so that

$$\int_{(\Omega_{n,t})^c} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) \leq e^{-\frac{1}{2}\theta^2 n} \int_{\Omega_{n,t}^c} e^{\theta B^1(n)} d\mu_0^S(B^1) \leq e^{-\frac{1}{2}\theta^2 n + \theta(t - mn)},$$

which, for $\theta > 0$, tends to zero as $n \rightarrow \infty$. It follows by the monotone convergence theorem and (4.24) that

$$\begin{aligned} 1 &= \int_{C_0([0, \infty) \rightarrow \mathbf{R})} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_{n,t}} M_\theta((\hat{T}B^1)(t) \wedge n) d\mu_0^S(B^1) = \int_{C_0([0, \infty) \rightarrow \mathbf{R})} M_\theta((\hat{T}B^1)(t)) d\mu_0^S(B^1) \\ &= \int_{C_0([0, \infty) \rightarrow \mathbf{R})} e^{\theta[t - m(\hat{T}B^1)(t)] - \frac{1}{2}\theta^2(\hat{T}B^1)(t)} d\mu_0^S(B^1), \end{aligned}$$

whence

$$\begin{aligned} e^{-\theta t} &= \int_{C_0([0, \infty) \rightarrow \mathbf{R})} e^{-\frac{1}{2}\theta(\theta + 2m)(\hat{T}B^1)(t)} d\mu_0^S(B^1) \\ &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{-\frac{1}{2}\theta(\theta + 2m)T(t)} d\nu_0(T). \end{aligned}$$

Taking $\theta = \sqrt{2\sigma + m^2} - m$ yields the result, showing Proposition 4.4. \square

This proposition implies that the characteristic function of the measure ν_0 is given by

$$\begin{aligned} e^{-tV(\rho)} &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{iT(t)\rho} d\nu_0(T), \quad \rho \in \mathbf{R}, \\ V(\rho) &= \frac{\sqrt{m^4 + 4\rho^2} - m^2}{\sqrt{2}[(m^2 + \sqrt{m^4 + 4\rho^2})^{1/2} + \sqrt{2}m]} - \frac{\sqrt{2}\rho}{(m^2 + \sqrt{m^4 + 4\rho^2})^{1/2}} i \\ &= \frac{2\sqrt{2}\rho^2}{[(m^2 + \sqrt{m^4 + 4\rho^2})^{1/2} + \sqrt{2}m](m^2 + \sqrt{m^4 + 4\rho^2})} \\ &\quad - \frac{\sqrt{2}\rho}{(m^2 + \sqrt{m^4 + 4\rho^2})^{1/2}} i. \end{aligned} \tag{4.26}$$

To see this, first analytically extend $\sqrt{2\sigma + m^2}$ to the right-half complex plane $z := \sigma + i\rho$, $\sigma > 0$, $\rho \in \mathbf{R}$, and next we have $V(-\rho) = \lim_{\sigma \rightarrow +0} \sqrt{2(\sigma + i\rho) + m^2} - m$, of which the right-hand side is to be calculated. Then (4.26) follows with ρ replaced by $-\rho$. [cf. Using a subordinator $T(t)$ slightly different from (4.24), in [I9-12, (4.18),

(4.19), (4.20), p.335] there are given a little different formulas corresponding to (4.25) and (4.26), $V(\rho)$. However, it contains an error; “ ρ^2 ” in the expression for $V(\rho)$ there should be replaced by “ $4\rho^2$ ”.]

Now we are in a position to give a path integral representation for $e^{-t[H^{(3)}-m]}g$.

Theorem 4.5.

$$(e^{-t[H^{(3)}-m]}g)(x) = \int \int_{\substack{C_x([0,\infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0,\infty) \rightarrow \mathbf{R})}} e^{-S^{(3)}(B,T;x,t)} g(B(T(t))) d\mu_x^1(B) d\nu_0(T), \quad (4.27)$$

$$\begin{aligned} S^{(3)}(B,T;x,t) &= i \int_0^{T(t)} A(B(s)) dB(s) + \frac{i}{2} \int_0^{T(t)} \operatorname{div} A(B(s)) ds + \int_0^t V(B(T(s))) ds, \\ &\equiv i \int_0^{T(t)} A(B(s)) \circ dB(s) + \int_0^t V(B(T(s))) ds, \end{aligned} \quad (4.28)$$

where μ_x^1 stands for the Wiener measure $\mu_x^m \equiv \mu_x$ on $C_x([0,\infty) \rightarrow \mathbf{R}^d)$ with characteristic function

$$\exp \left[-t \frac{|\xi|^2}{2} \right] = \int_{C_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{i(B(t)-x) \cdot \xi} d\mu_x^1(B), \quad (4.29)$$

with mass taken as $m = 1$.

Remark. We note that for every pair $(B,T) \in C_x([0,\infty) \rightarrow \mathbf{R}^d) \times D_0([0,\infty) \rightarrow \mathbf{R})$ the path $B(T(s))$ in this theorem belongs to $D_x([0,\infty) \rightarrow \mathbf{R}^d)$, because it is right-continuous in $s \in [0,\infty)$ and has left-hand limit. The characteristic function of the product $\mu_x^1 \times \nu_0$ of the probability measures is calculated with (4.29) as

$$\begin{aligned} &\int \int_{\substack{C_x([0,\infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0,\infty) \rightarrow \mathbf{R})}} e^{i(B(T(t))-x) \cdot \xi} d\mu_x^1(B) d\nu_0(T) \\ &= \int_{D_0([0,\infty) \rightarrow \mathbf{R})} \exp \left[-T(t) \frac{|\xi|^2}{2} \right] d\nu_0(T) = e^{-t[\sqrt{|\xi|^2+m^2}-m]}, \end{aligned} \quad (4.30)$$

thus coinciding with (4.4), the characteristic function of the measure λ_x . This implies that these two processes on the two different probability spaces $(C_x([0,\infty) \rightarrow \mathbf{R}^d) \times D_0([0,\infty) \rightarrow \mathbf{R}), \mu_x^1 \times \nu_0)$ and $(D_x([0,\infty) \rightarrow \mathbf{R}^d), \lambda_x)$ are identical in law, i.e. have the same finite-dimensional distributions, in fact, for $0 < t_1 < t_2 < \dots < t_n < \infty$ and $n = 1, 2, 3, \dots$,

$$\begin{aligned} &\int \int_{\substack{C_x([0,\infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0,\infty) \rightarrow \mathbf{R})}} e^{i[(B(T(t_1))-x) \cdot \xi_1 + (B(T(t_2))-x) \cdot \xi_2 + \dots + (B(T(t_n))-x) \cdot \xi_n]} d\mu_x^1(B) d\nu_0(T) \\ &= \int_{D_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{i[(X(t_1)-x) \cdot \xi_1 + (X(t_2)-x) \cdot \xi_2 + \dots + (X(t_n)-x) \cdot \xi_n]} d\lambda_x(X) \\ &= e^{-t_1[\sqrt{|\xi_1+\xi_2+\dots+\xi_n|^2+m^2}-m]} e^{-(t_2-t_1)[\sqrt{|\xi_2+\dots+\xi_n|^2+m^2}-m]} \dots \\ &\quad \times e^{-(t_n-t_{n-1})[\sqrt{|\xi_n|^2+m^2}-m]}. \end{aligned}$$

Therefore the former process may also be considered basically a Lévy process, but it is not clear whether one can rewrite the right-hand side of (4.27) as a process on

the probability space $(D_x([0, \infty) \rightarrow \mathbf{R}^d), \lambda_x)$, replacing the function $S^{(3)}(B, T; x, t)$ of B, T, x, t in (4.28) with some function else $S^{(3)}(X; x, t)$ of X, x, t appropriately written in terms of the Lévy space path X in $D_x([0, \infty) \rightarrow \mathbf{R}^d)$.

Proof of Theorem 4.5. We give only a sketch. The detail is referred to [IHL01-12]. We use Proposition 4.4 and the Feynman–Kac–Itô formula (4.2). Note that $H_A^{(3)} = \sqrt{2H_A^{NR} + m^2}$ where $H_A^{NR} = \frac{1}{2}(-i\nabla - A(x))^2$ is the magnetic nonrelativistic Schrödinger operator with mass 1. By the spectral theorem for the nonnegative self-adjoint operator H_A^{NR} , we have $H_A^{NR} = \int_{\text{Spec}(H_A^{NR})} \sigma dE(\sigma)$, where $E(\cdot)$ is the spectral measure associated with H_A^{NR} . Then for $f, g \in L^2(\mathbf{R}^d)$

$$(f, e^{-t[H_A^{(3)} - m]} g) = \int_{\text{Spec}(H_A^{NR})} e^{-t[\sqrt{2\sigma + m^2} - m]} (f, dE(\sigma)g).$$

Here we are using the *physicist's inner product* (f, g) , which is anti-linear in f and linear in g . By Proposition 4.4 and again by the spectral theorem,

$$\begin{aligned} (f, e^{-t[H_A^{(3)} - m]} g) &= \int_{\text{Spec}(H_A^{NR})} \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{-T(t)\sigma} d\nu_0(T) (f, dE(\sigma)g) \\ &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} (f, e^{-T(t)H_A^{NR}} g) d\nu_0(T). \end{aligned}$$

Applying the Feynman–Kac–Itô formula (4.2) (for the case $V = 0$) to $e^{-T(t)H_A^{NR}} g$ in the third member, we have

$$\begin{aligned} &(f, e^{-t[H_A^{(3)} - m]} g) \\ &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} d\nu_0(T) \int_{\mathbf{R}^d} dx \overline{f(B(0))} \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-i \int_0^{T(t)} A(B(s)) \circ dB(s)} g(B(T(t))) d\mu_x^1(B) \\ &= \int_{\mathbf{R}^d} dx \overline{f(x)} \int_{\substack{C_x([0, \infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0, \infty) \rightarrow \mathbf{R})}} e^{-i \int_0^{T(t)} A(B(s)) \circ dB(s)} g(B(T(t))) d\mu_x^1(B) d\nu_0(T), \end{aligned}$$

where note $B(0) = x$. This proves the assertion when $V = 0$.

When $V \neq 0$, with partition of $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$, $t_j - t_{j-1} = t/n$, we can express $e^{-t[H_A^{(3)} - m]} g = e^{-t[(H_A^{(3)} - m) + V]}$ by the Trotter–Kato formula (4.23). Rewrite the product of these n operators by path integral with respect to the product of two probability measures $\nu_0(T) \cdot \mu_x^1(B)$ and note that $T(0) = T(t_0) = 0$, $B(0) = B(T(t_0)) = x$, then we have

$$\begin{aligned} &(f, (e^{-(t/n)[H_A^{(3)} - m]} e^{-(t/n)V})^n g) \\ &= \int_{\mathbf{R}^d} dx \int_{D_0([0, \infty) \rightarrow \mathbf{R})} d\nu_0(T) \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} \overline{f(B(0))} \\ &\quad \times e^{-i \sum_{j=1}^n \int_{T(t_{j-1})}^{T(t_j)} A(B(s)) \circ dB(s)} e^{-\sum_{j=1}^n V(B(T(t_j))) \frac{t}{n}} g(B(t_n)) d\mu_x^1(B). \end{aligned}$$

We see that, as $n \rightarrow \infty$, the left-hand side converges to $(f, e^{-t[H_A^{(3)} - m]} g)$, and the Lebesgue theorem shows the right-hand side converges as integral by the product mea-

sure $dx \times \nu_0(T) \times \mu_x^1(B)$, so that we obtain

$$\begin{aligned} & (f, (e^{-t[H^{(3)}-m]}g)) \\ &= \int_{\mathbf{R}^d} dx \overline{f(x)} \int_{\substack{C_x([0,\infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0,\infty) \rightarrow \mathbf{R})}} e^{-S^{(3)}(B,T;x,t)} g(B(T(t))) d\mu_x^1(B) d\nu_0(T). \end{aligned}$$

Hence or similarly we can also get (4.27)/(4.28) with f removed in the above inner products. \square

4.2 Heuristic derivation of path integral formulas

After a brief introduction to *path integral*, we discuss, for the solution of the imaginary-time magnetic relativistic Schrödinger equation (4.3), how to heuristically derive its path integral formulas (4.9)/(4.10) in Theorem 4.1, (4.20)/(4.21) in Theorem 4.3 and (4.27)/(4.28) in Theorem 4.5.

4.2.1. What is *path integral* ?

It is a fabulous technique invented by Feynman in his Princeton 1942 thesis (see [Fey2-05]) and his 1948 paper [Fey1-48] to give alternative formulation of quantum mechanics. Its like has never been made before or since. In fact, though it is not mathematically rigorous, because of the universality of its idea, it has now come to prevail over all the domains in quantum physics. It is interesting to note, as he himself wrote in [48], that he came to the idea, “suggested by some of Dirac’s remarks ([Di1-33, 2-35], [Di3-45]) concerning the relation of classical action to quantum mechanics.” It is a special kind of *functional integral* like

$$\int e^{\frac{i}{\hbar}S(X)} \mathcal{D}[X] \quad (4.31)$$

on space of paths $X : [0, t] \ni s \mapsto X(s) \in \mathbf{R}^d$ with respect to a ‘measure’ $\mathcal{D}[X]$ on the space of these paths, where we are restoring the physical constant $\hbar = \frac{h}{2\pi}$ ($h > 0$: Planck’s constant). $S(X)$ is time integral of the *Lagrangian* $L(X(s), \dot{X}(s))$ where $\dot{X}(s) = \frac{d}{ds}X(s)$:

$$S(X) = \int_0^t L(X(s), \dot{X}(s)) ds,$$

which is an important quantity in classical mechanics, called *action* along the path X , having physical dimension of Planck’s constant h so that $\frac{S(X)}{\hbar}$ becomes dimensionless.

We have in mind the nonrelativistic-quantum-mechanical motion of a particle in space \mathbf{R}^d under influence of the scalar potential $V(x)$. In the previous sections, we let the particle have the special mass $m = 1$, but in this section, for a while, assume it to have general mass $m > 0$ so that we can see where m appears in the following description of its dynamics. Thus consider the Cauchy problem for the nonrelativistic Schrödinger equation for this particle with initial data $\psi(x, 0) = f(x)$:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[-\frac{\hbar^2}{2m} \Delta + V(x) \right] \psi(x, t), \quad t > 0, \quad x \in \mathbf{R}^d. \quad (4.32)$$

The solution is expressed as

$$\psi(x, t) = \int K(x, t; y, 0) f(y) dy,$$

where $K(x, t; y, 0)$ is integral kernel, called *fundamental solution* or *propagator*.

Feynman writes down this important quantity $K(x, t; s, y)$ as an ‘integral’

$$K(x, t; y, 0) = \int_{\{X: X(0)=y, X(t)=x\}} e^{\frac{i}{\hbar} S(X)} \mathcal{D}[X], \quad (4.33)$$

where in the present case $L(X(s), \dot{X}(s)) = \frac{m}{2} \dot{X}(s)^2 - V(X(s))$, so that $S(X)$ is given by

$$S(X) = \int_0^t \left[\frac{m}{2} \dot{X}(s)^2 - V(X(s)) \right] ds. \quad (4.34)$$

$\mathcal{D}[X]$ stands for a uniform ‘measure’, if it exists, on the space of paths $X(\cdot)$ starting from position y in space at time 0 to arrive at position x in space at time t , *formally* to be given by the infinite product of continuously-many number of the Lebesgue measures $dX(\tau)$ on space \mathbf{R}^d for each individual τ :

$$\mathcal{D}[X] := \text{“constant”} \times \prod_{0 < \tau < t} dX(\tau).$$

Here the “constant” should be something like $\left(\frac{im}{2\pi\hbar(\delta t)}\right)^{\frac{d}{2}(\frac{t}{\delta t}-1)}$ with δt being some infinitesimal quantity of time, if one dares to try to write it, wondering what it means at all, but one may infer something from around (4.37) below. The right-hand side of (4.33) is what is called *Feynman path integral* or, nowadays simply, *path integral*.

To explain this, Feynman put the following *Two Postulates* which turn out to be equivalent to get the above expression (4.33) for $K(x, t; y, 0)$, so that, for $f, g \in L^2(\mathbf{R}^d)$,

$$(f, \psi(\cdot, t)) = (f, e^{-\frac{it}{\hbar}[-\frac{\hbar^2}{2m}\Delta + V]} g) = \int \int \overline{f(x)} K(x, t; y, 0) g(y) dx dy.$$

Feynman’s Two Postulates

(i) $K(x, t; y, 0)$ is the *total probability amplitude* for the event that the particle starts from position y at time 0 and arrives at position x at time t . If $\varphi[X]$ stands for the *probability amplitude* for the event that it does this motion *along each individual path* $X(\cdot)$, $K(x, t; y, 0)$ is the sum of the $\varphi[X]$ over all these paths $X(\cdot)$:

$$K(x, t; y, 0) = \sum_{\{X: X(0)=y, X(t)=x\}} \varphi[X]. \quad (4.35)$$

(ii) The contribution $\varphi[X]$ from each $X(\cdot)$ to the total probability amplitude $K(x, t; y, 0)$ is given by

$$\varphi[X] = C e^{\frac{i}{\hbar} S(X)}, \quad (4.36)$$

where C is a constant independent of path X .

These two postulates can be paraphrased: In quantum mechanics there rules the Principle of Democracy that each individual path $X(\cdot)$ contributes to the total probability amplitude $K(x, t; y, 0)$ with *equal weight* (*absolute value* in mathematics) and its personality is expressed by its *phase* (*argument* in mathematics).

In this respect, in classical mechanics there does not rule Principle of Democracy, because the particle takes a *particular path* between two space-time points $(y, 0)$ and (x, t) which makes the action $S(X)$ stationary, called *classical trajectory*. It is the path

determined by Euler–Lagrange equation or, in the present case, Newton’s equation of motion : $m \frac{d^2}{ds^2} X(s) = -\nabla V(X(s))$.

The most important characteristic feature of these postulates lies in equation (4.36), which says that the amplitude $\varphi[X]$ is propotional to the phase $e^{\frac{i}{\hbar}S(X)}$. The phrase “propotional to” is that which Feynman determined to substitute for what Dirac had meant by the phrase “analogous to” in [Di1-33, 2-35], [Di3-45] far before Feynman, by showing after his own analysis and deliberation that *indeed this exponential function could be used in this manner directly* (see Preface of [FeyHi-65]).

In classical mechanical circumstances, \hbar is so small compared with other physical quantities that one may ignore and think of it as zero. The amazing thing is that this ‘integral’ (4.33) can let us see how the transition is going to classical mechanics as \hbar tends to zero. Namely, when \hbar tends to zero, if the stationary phase method should be valid for this ‘integral’ (4.33), then the ‘integral’ would turn out to receive the most crucial contribution from the path which makes the action $S(X)$ stationary, i.e. the classical trajectory (mentioned above) and its neighboring paths.

4.2.2. How to make it mathematics ?

Here we refer, among others, only to two methods; one is by finite-dimensional approximation, and the other by imaginary-time path integral. In fact, it is by the first method that Feynman himself confirmed his idea of path integral. He calculated $K(x, t; y, 0)$ by time-sliced approximation, making partition of the time interval $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$, $(t_j - t_{j-1} = t/n)$, $x_j := X(t_j)$, $x_0 = X(0) = y$, $x_n = X(t) = x$, as the limit of

$$K_n(x, t; y, 0) := \frac{\int_{(\mathbf{R}^d)^{n-1}} \exp\left\{\frac{it}{\hbar n} \sum_{j=0}^{n-1} \left[\frac{m}{2} \left(\frac{x_{j+1} - x_j}{t/n}\right)^2 - V(x_j)\right]\right\} dx_1 \cdots dx_{n-1}}{\int_{(\mathbf{R}^d)^n} \exp\left\{\frac{it}{\hbar n} \sum_{j=0}^{n-1} \frac{m}{2} \left(\frac{x_{j+1} - x_j}{t/n}\right)^2\right\} dx_1 \cdots dx_{n-1} dx_n} \quad (4.37)$$

as $n \rightarrow \infty$, to ascertain it to satisfy the Schrödinger equation (4.32). Note that the denominator of the right-hand side of (4.37) is equal to $\left(\frac{2\pi i \hbar \frac{t}{n}}{m}\right)^{\frac{d}{2}n}$.

The second method is the one which the present article is mainly concerning. We note with (4.33) that the solution $\psi(x, t)$ of the Schrödinger equation (4.32) turns out to be given by a heuristic path integral

$$\psi(x, t) = \int_{\mathbf{R}^d} K(x, t; y, 0) f(y) dy = \int_{\{X: X(t)=x\}} e^{\frac{i}{\hbar}S(X)} f(X(0)) \mathcal{D}[X]. \quad (4.38)$$

However, one should know that the ‘measure’ $\mathcal{D}[X]$ itself in general does not exist in this situation as a countably additive measure. Therefore we cannot go further. But if we rotate everthing by -90° : $t \rightarrow -it$ (real-time t to imaginary-time $-it$) in complex t -plane (see Figure 1), i.e. if we go from our *Minkowski* space-time to *Euclidian* space-time, the situation will change. Before actually doing it, for simplify put $\hbar = 1$. Then, as our rotation also converts ds to $-ids$, so does it $\dot{X}(s) = \frac{dX(s)}{ds}$ to $i\dot{X}(s) = \frac{dX(s)}{-ids}$ [where we don’t mind thinking of “ $X(-is)$ ” as $X(s)$ again], so that $iS(X)$, the action $S(X)$ in (4.3) mutiplied by $i = \sqrt{-1}$, is converted to time integral of the *Hamiltonian* : $-\int_0^t [\frac{m}{2} \dot{X}(s)^2 + V(X(s))] ds$. Simultaneously, our (real-time) Schrödinger equation (4.32) is converted to the imaginary-time Schrödinger equation, i.e. heat equation [where writing $u(x, t)$ for $\psi(x, -it)$]:

$$\frac{\partial}{\partial t} u(x, t) = \left[\frac{1}{2m} \Delta - V(x) \right] u(x, t), \quad t > 0, \quad x \in \mathbf{R}^d. \quad (4.39)$$

Now we are going to get to the so-called *Feynman–Kac formula*. To this end, we replace the paths used so far by the time-reversed paths $X_0(s) := X(t-s)$, $0 \leq s \leq t$, so that $X_0(0) = X(t) = x$, $X_0(y) = X(0) = y$. Then $K(x, t; y, 0)$ is changed to

$$K^E(x, t; y, 0) := \int_{\{X_0: X_0(0)=x, X_0(t)=y\}} e^{-\int_0^t [\frac{m}{2} \dot{X}_0(s)^2 + V(X_0(s))] ds} \mathcal{D}[X_0]. \quad (4.40)$$

where the superscript “ E ” is attributed to “Euclidian”, and $K^E(x, t; y, 0)$ should become the integral kernel for the heat equation (4.39). In passing we quickly insert here: if one were to follow the first method by Feynman as (4.37), one could also define $K^E(x, t; y, 0)$ as the limit as $n \rightarrow \infty$ of

$$K_n^E(x, t; y, 0) := \frac{\int_{(\mathbf{R}^d)^{n-1}} \exp\left\{-\frac{t}{n} \sum_{j=0}^{n-1} \left[\frac{m}{2} \left(\frac{x_{j+1}-x_j}{t/n}\right)^2 + V(x_j)\right]\right\} dx_1 \cdots dx_{n-1}}{\int_{(\mathbf{R}^d)^n} \exp\left\{-\frac{t}{n} \sum_{j=0}^{n-1} \left[\frac{m}{2} \left(\frac{x_{j+1}-x_j}{t/n}\right)^2\right]\right\} dx_1 \cdots dx_{n-1} dx_n}, \quad (4.41)$$

where $t_j - t_{j-1} = t/n$ ($j = 1, 2, \dots, n$); $x_j = X_0(t_j)$, $x_0 = X_0(0) = x$, $x_n = X_0(t) = y$.

We infer from (4.40) that the solution of the Cauchy problem for (4.39) with initial data $u(x, 0) = g(x)$ should be given by the following path integral

$$\begin{aligned} u(x, t) &= \int_{\mathbf{R}^d} K^E(x, t; y, 0) g(y) dy \\ &= \int_{\{X_0: X_0(0)=x\}} e^{-\int_0^t [\frac{m}{2} \dot{X}_0(s)^2 + V(X_0(s))] ds} g(X_0(t)) \mathcal{D}[X_0]. \end{aligned} \quad (4.42)$$

Here we have tacitly identified the two ‘integrals’:

$$\int_{\mathbf{R}^d} dy \int_{\{X_0: X_0(0)=x, X_0(t)=y\}} \cdots \mathcal{D}[X_0] \sim \int_{\{X_0: X_0(0)=x\}} \cdots \mathcal{D}[X_0].$$

Needless to say, when the scalar potential $V(x)$ is absent, $K^E(x, t; y, 0)$ becomes the heat kernel $\left(\frac{2\pi t}{m}\right)^{-d/2} e^{-\frac{m}{2t}|x-y|^2}$, which is obtained as the inverse Fourier transform of the left-hand side of (4.29), or by calculating the integrals (4.41) and taking the limit $n \rightarrow \infty$. Note that the denominator of (4.41) is equal to $\left(\frac{2\pi t}{m}\right)^{\frac{d}{2} \cdot n}$. By (4.40) we also see it have the following heuristic expression on the right-hand side

$$\frac{e^{-\frac{m}{2t}|x-y|^2}}{\left(\frac{2\pi t}{m}\right)^{d/2}} = \int_{\{X_0: X_0(0)=x, X_0(t)=y\}} e^{-\int_0^t \frac{m}{2} \dot{X}_0(s)^2 ds} \mathcal{D}[X_0].$$

Remarkable is that Wiener, already around 1923, had constructed, for each individual $x \in \mathbf{R}^d$, a countably additive measure μ_x with $m = 1$ (but of course valid for every $m > 0$) on the space $C_x := C_x([0, \infty) \rightarrow \mathbf{R}^d)$ of the continuous paths (*Brownian motions*) $B : [0, \infty) \ni s \mapsto B(s) \in \mathbf{R}^d$ starting from $B(0) = x$ at time $t = 0$. Further μ_x is a probability measure on C_x with characteristic function (4.29), and now is called *Wiener measure*.

Around 1947, Kac, who had been at Cornell University as Feynman and heard his lecture at the Physics Colloquium, struck upon the very idea of using the Wiener measure to represent the solution $u(x, t)$ of the Cauchy problem for the heat equation (4.39) (with $m = 1$) with initial data $u(x, 0) = g(x)$ as a first mathematical rigorous, *genuine functional integral*

$$u(x, t) = \int K^E(x, t; y, 0) g(y) dy = \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-\int_0^t V(B(s)) ds} g(B(t)) d\mu_x(B), \quad (4.43)$$

the same formula as (4.1) already mentioned at the top of this section. This is the *Feynman–Kac formula* [Kac-66/80] mentioned in advance. Thus, identify the path $X_0(\cdot)$ appearing on the right-hand side of (4.40)/(4.42) with the continuous path $B(\cdot)$ in the space $C_x([0, \infty) \rightarrow \mathbf{R}^d)$, then the Wiener measure μ_x turns out to be constructed from the factor “ $e^{-\int_0^t \frac{m}{2} \dot{B}(s)^2 ds} \mathcal{D}[B]$ ” (with $m = 1$) on the right-hand side of (4.40)/(4.42).

4.2.3. The case for relativistic Schrödinger equation.

We begin with the relativistic Schrödinger equation for a relativistic particle of mass m with positive energy in an electromagnetic field:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = [H - m] \psi(x, t), \quad t > 0, \quad x \in \mathbf{R}^d, \quad (4.44)$$

where H is one of the relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ corresponding to the classical symbol $\sqrt{(\xi - A(x))^2 + m^2} + V(x)$. This equation (4.44) was already briefly mentioned at the top of Section 4.1. However, here for the moment we go with the quantization “ $\xi \rightarrow -i\hbar \nabla$ ” with \hbar recovered, but not “ $\xi \rightarrow -i\nabla$ ” used there. In this case it is more appropriate to use the method of *phase space path integral* or *Hamiltonian path integral* (Feynman [Fey-65, p.125], Garrod [G-66], Mizrahi [M-78]):

$$\int e^{\frac{i}{\hbar} S(\Xi, X)} \mathcal{D}[\Xi] \mathcal{D}[X] \quad (4.45)$$

with a ‘measure’ $\mathcal{D}[\Xi] \mathcal{D}[X]$ on the space of phase space paths (Ξ, X) , pairs of momentum path $\Xi(s)$ and position path $X(s)$ on the phase space $\mathbf{R}^d \times \mathbf{R}^d$, and with *action* written with each pair $(\Xi(s), X(s))$. Then, under this circumstance, the previous path integral (4.31), (4.33) in the nonrelativistic case is also called *configuration path integral*.

The solution $\psi(x, t)$ of the Cauchy problem for (4.44) with initial data $\psi(x, 0) = f(x)$ can be written as $\psi(x, t) = \int K(x, t; y, 0) f(y) dy$ with integral kernel $K(x, t; y, 0)$ called *fundamental solution* or *propagator*. Then the method of phase space path integral or Hamiltonian path integral assumes $K(x, t; y, 0)$ to have the following path integral representation:

$$K(x, t; y, 0) = \int_{\{(\Xi, X); X(0)=y, X(t)=x, \Xi: \text{arbitrary}\}} e^{\frac{i}{\hbar} S(\Xi(s), X(s))} \mathcal{D}[\Xi] \mathcal{D}[X]. \quad (4.46)$$

Here the *action* $S(\Xi, X)$ along the phase space path (Ξ, X) is given by

$$S(\Xi, X) = \int_0^t \left[\Xi(s) \cdot \dot{X}(s) - \left(\sqrt{(\Xi(s) - A(X(s)))^2 + m^2} - m + V(X(s)) \right) \right] ds, \quad (4.47)$$

where $\dot{X}(s) = \frac{d}{ds} X(s)$ in the same way as in the nonrelativistic case (4.34). $\mathcal{D}[\Xi] \mathcal{D}[X]$ is a uniform ‘measure’, if it exists, on the space of phase space paths $(\Xi, X) : [0, t] \ni s \mapsto (\Xi(s), X(s)) \in \mathbf{R}^d \times \mathbf{R}^d$ with $X(0) = y, X(t) = x$, but Ξ being unrestricted and so arbitrary, *formally* to be given by the infinite product of continuously-many number of the Lebesgue measures $d\Xi(\tau) dX(\tau)$ (precisely, divided by $(2\pi)^d$) on phase space $\mathbf{R}^{2d} = \mathbf{R}^d \times \mathbf{R}^d$ for each individual τ :

$$\mathcal{D}[\Xi] \mathcal{D}[X] := \prod_{0 < \tau < t} \frac{d\Xi(\tau) dX(\tau)}{(2\pi)^d}.$$

In this ‘integral’ (4.46) we make the transform of variables (paths): $\Xi'(s) = \Xi(s) - A(X(s))$ and $X'(s) = X(s)$, where we note the *formal* Jacobi determinant $\frac{\partial(\Xi(s), X(s))}{\partial(\Xi'(s), X'(s))}$ of this transform is 1. Write $\Xi(s)$ for $\Xi'(s)$ and $X(s)$ for $X'(s)$ again, then (4.46) becomes

$$K(x, t; y, 0) = \int_{\{(\Xi, X); X(0)=y, X(t)=x, \Xi: \text{arbitrary}\}} \times e^{\frac{i}{\hbar} \int_0^t \left[(\Xi(s) + A(X(s))) \cdot \dot{X}(s) - \left(\sqrt{\Xi(s)^2 + m^2} - m + V(X(s)) \right) \right] ds} \mathcal{D}[\Xi] \mathcal{D}[X]. \quad (4.48)$$

We want to find a path integral formula for the solution of the imaginary-time magnetic relativistic Schrödinger equation (4.3). For simplicity we put $\hbar = 1$ as before. We go from real time t to imaginary time $-it$. This procedure also converts ds to $-ids$ and so $\dot{X}(s) = \frac{d}{ds}X(s)$ to $i\dot{X}(s) = \frac{dX(s)}{-ids}$ [where we don’t mind thinking of “ $\Xi(-is)$, $X(-is)$ ” as $\Xi(s)$, $X(s)$, respectively, again]. Then we replace the phase space paths used so far by the time-reversed phase space paths: $X_0(s) := X(t-s)$, $\Xi_0(s) := \Xi(t-s)$, $0 \leq s \leq t$, so that $X_0(0) = X(t) = x$, $X_0(y) = X(0) = y$. As a result, (4.46) is changed to

$$K^E(x, t; y, 0) := \int_{\{(\Xi_0, X_0); X_0(0)=x, X_0(t)=y, \Xi_0: \text{arbitrary}\}} \times e^{\int_0^t \left[i(\Xi_0(s) + A(X_0(s))) \cdot \dot{X}_0(s) - \left(\sqrt{\Xi_0(s)^2 + m^2} - m + V(X_0(s)) \right) \right] ds} \mathcal{D}[\Xi_0] \mathcal{D}[X_0]. \quad (4.49)$$

At this final stage we rewrite $\Xi_0(s)$, $X_0(s)$ as $\Xi(s)$, $X(s)$ again. Thus we have heuristically arrived, for the solution $u(x, t)$ of the Cauchy problem for (4.3) with initial data $u(x, 0) = g(x)$, at the following path integral representation:

$$\begin{aligned} u(x, t) &= (e^{-t[H-m]}g)(x) = \int K^E(x, t; y, 0)g(y)dy \\ &= \int_{\{X(0)=x\}} e^{\int_0^t \left[i(\Xi(s) + A(X(s))) \cdot \dot{X}(s) - \left(\sqrt{\Xi(s)^2 + m^2} - m + V(X(s)) \right) \right] ds} g(X(t)) \mathcal{D}[\Xi] \mathcal{D}[X] \\ &= \int_{\{X(0)=x\}} e^{\int_0^t \left[iA(X(s)) \cdot \dot{X}(s) - V(X(s)) \right] ds} \\ &\quad \times e^{\int_0^t \left[i\Xi(s) \cdot \dot{X}(s) - \left(\sqrt{\Xi(s)^2 + m^2} - m \right) \right] ds} g(X(t)) \mathcal{D}[\Xi] \mathcal{D}[X]. \end{aligned} \quad (4.50)$$

Now we ask whether our path integral formulas, (4.9)/(4.10) in Theorem 4.1, (4.20)/(4.21) in Theorem 4.3 and (4.27)/(4.28) in Theorem 4.5, can be well derived or at least well inferred from this formal expression of ‘integral’ (4.50). First of all, if both the vector and scalar potentials $A(x)$ and $V(x)$ are absent, (4.50) is reduced to

$$\begin{aligned} u(x, t) &= (e^{-t[\sqrt{-\Delta+m^2}-m]}g)(x) = \int k_0(x-y, t)g(y)dy \\ &= \int_{\{X(0)=x\}} e^{\int_0^t \left[i\Xi(s) \cdot \dot{X}(s) - \left(\sqrt{\Xi(s)^2 + m^2} - m \right) \right] ds} g(X(t)) \mathcal{D}[\Xi] \mathcal{D}[X], \end{aligned} \quad (4.51)$$

where $k_0(x-y, t)$ is the integral kernel of the semigroup $e^{-t[\sqrt{-\Delta+m^2}-m]}$ in (3.3), and,

similarly to the nonrelativistic case, we have identified the two ‘integrals’:

$$\begin{aligned} & \int_{\mathbf{R}^d} dy \int_{\{(\Xi, X); X(0)=x, X(t)=y, \Xi: \text{arbitrary}\}} \cdots \mathcal{D}[\Xi] \mathcal{D}[X] \\ & \sim \int_{\{(\Xi, X); X(0)=x, \Xi: \text{arbitrary}\}} \cdots \mathcal{D}[\Xi] \mathcal{D}[X]. \end{aligned}$$

Noting that the second and/or third member of (4.50) is equal to

$$\int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} g(X(t)) d\lambda_x(X),$$

we see that the factor

$$\exp \left\{ \int_0^t \left[i\Xi(s) \cdot \dot{X}(s) - \left(\sqrt{\Xi(s)^2 + m^2} - m \right) \right] ds \right\} \mathcal{D}[\Xi] \mathcal{D}[X] \quad (4.52)$$

turns out to be identified with the probability measure λ_x , (4.4) introduced in Section 4.1, connected with the Lévy process concerned. Next, we shall see that, since there is no problem for the factor $e^{-\int_0^t V(X(s))ds}$, the problem lies only in how to interpret and understand the factor

$$e^{i \int_0^t A(X(s)) \cdot \dot{X}(s) ds} = \prod_{j=1}^n e^{i \int_{t_{j-1}}^{t_j} A(X(s)) \cdot \dot{X}(s) ds},$$

when dividing the time interval $[0, t]$ into n equal small subintervals $[t_0, t_1], \dots, [t_{n-1}, t_n]$ with $t_j = jt/n$, $j = 0, 1, 2, \dots, n$, in fact, whether, for small interval $[t_{j-1}, t_j]$ or large n , the factor $e^{i \int_{t_{j-1}}^{t_j} A(X(s)) \cdot \dot{X}(s) ds}$ can allow a good approximation to be suggested by the obtained path integral formulas for $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$.

(1) First we consider the case for $H^{(1)}$ by approximating the factor

$$\exp \left[i \int_{t_{j-1}}^{t_j} A(X(s)) \cdot \dot{X}(s) ds \right] \quad \text{by} \quad \exp \left[i A \left(\frac{X(t_{j-1}) + X(t_j)}{2} \right) \cdot (X(t_j) - X(t_{j-1})) \right]$$

on each subinterval $[t_{j-1}, t_j]$ (“midpoint prescription”). Then the last member of (4.15) is expected to be the limit $n \rightarrow \infty$ of

$$\begin{aligned} & \overbrace{\int_{\mathbf{R}^{2d}} \cdots \int_{\mathbf{R}^{2d}}}^{n \text{ times}} \left(e^{i \sum_{j=1}^n \left(\Xi(t_j) + A \left(\frac{X(t_{j-1}) + X(t_j)}{2} \right) \cdot (X(t_j) - X(t_{j-1})) \right)} \right. \\ & \times e^{-\sum_{j=1}^n \left[\sqrt{\xi_j^2 + m^2} - m + V \left(\frac{X(t_{j-1}) + X(t_j)}{2} \right) \right] \frac{t}{n}} \Big) g(X(t_n)) \prod_{j=1}^n \frac{d\Xi(t_j) dX(t_j)}{(2\pi)^d} \\ & = \overbrace{\int_{\mathbf{R}^{2d}} \cdots \int_{\mathbf{R}^{2d}}}^{n \text{ times}} \left(e^{i \sum_{j=1}^n \left[(\Xi(t_j) \cdot (X(t_j) - X(t_{j-1})) - (\sqrt{\xi_j^2 + m^2} - m) \frac{t}{n}) \right]} \right. \\ & \times e^{i \sum_{l=1}^n \left[A \left(\frac{X(t_{l-1}) + X(t_l)}{2} \right) \cdot (X(t_j) - X(t_{j-1})) - V \left(\frac{X(t_{l-1}) + X(t_l)}{2} \right) \frac{t}{n} \right]} \Big) \\ & \times g(X(t_n)) \prod_{j=1}^n \frac{d\Xi(t_{j-1}) dX(t_{j-1})}{(2\pi)^d}, \quad X(0) = X(t_0) = x. \quad (4.53) \end{aligned}$$

Then putting $\xi_j = \Xi(t_j)$, $x_j = X(t_j)$ makes (4.50) equal to

$$\begin{aligned}
& \overbrace{\int_{\mathbf{R}^{2d}} \cdots \int_{\mathbf{R}^{2d}}}^{n \text{ times}} \prod_{j=1}^n e^{i(x_j - x_{j-1}) \cdot \xi_j} e^{-[\sqrt{\xi_j^2 + m^2} - m] \frac{t}{k}} \\
& \times \exp \left\{ i \sum_{l=1}^n \left[A \left(\frac{x_{l-1} + x_l}{2} \right) \cdot (x_l - x_{l-1}) - V \left(\frac{x_{l-1} + x_l}{2} \right) \frac{t}{n} \right] \right\} \\
& \times g(x_n) \prod_{j=1}^n \frac{d\xi_j dx_j}{(2\pi)^d}, \quad x_0 = x.
\end{aligned} \tag{4.54}$$

Performing all the $d\xi_j$ integrals yields

$$\begin{aligned}
& \overbrace{\int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d}}^{n \text{ times}} k_0(x_0 - x_1, t/n) k_0(x_1 - x_2, t/n) \cdots k_0(x_{n-1} - x_n, t/n) \\
& \times \exp \left\{ - \sum_{l=1}^n \left[i A \left(\frac{x_{l-1} + x_l}{2} \right) \cdot (x_{l-1} - x_l) + V \left(\frac{x_{l-1} + x_l}{2} \right) \frac{t}{n} \right] \right\} \\
& \times g(x_n) dx_1 \cdots dx_n, \quad x_0 = x,
\end{aligned} \tag{4.55}$$

where $k_0(x, t)$ is the integral kernel of $e^{-t[\sqrt{-\Delta + m^2} - m]}$ in (3.3). Note that (4.55) is the same as the second member of (4.15), which was shown in Proposition 4.2 to converge to $e^{-t[H^{(1)} - m]}g$. Therefore we may think that the expression (4.9) with (4.10) is heuristically connected with (4.50) in the limit $n \rightarrow \infty$ of the expression (4.55) which should be the path integral formula for $e^{-t[H^{(1)} - m]}g$ in Theorem 4.1.

(2) Next we consider the case for $H^{(2)}$ by approximating the factor

$$\exp \left[i \int_{t_{j-1}}^{t_j} A(X(s)) \cdot \dot{X}(s) ds \right]$$

by

$$\exp \left[i \int_0^1 A((1 - \theta)X(t_j) + \theta X(t_{j-1})) \cdot (X(t_j) - X(t_{j-1})) d\theta \right]$$

on each subinterval $[t_{j-1}, t_j]$. The same arguments as in (1) above will show the expression (4.20) with (4.21) is also heuristically connected with (4.50), leading to the path integral formula (4.20) with (4.21) for $e^{-t[H^{(2)} - m]}g$ in Theorem 4.3.

(3) Finally, we come to the case for $H^{(3)}$. Indeed, (4.27)/(4.28) is a mathematically rigorous, beautiful path integral but it does not seem to be one which can be heuristically deduced from the formal expression of ‘integral’ (4.50). We could not think the factor $\exp \left[i \int_{t_{j-1}}^{t_j} A(X(s)) \cdot \dot{X}(s) ds \right]$ can allow a good approximation to be suggested by (4.27)/(4.28) for $H^{(3)}$. It is because $H_A^{(3)}$ does not seem to be so explicitly well expressed by a pseudo-differential operator defined through a certain *tractable symbol* as $H_A^{(1)}$ and $H_A^{(2)}$.

5 Summary

Finally, we will collect here, as summary, the three path integral representation formulas in Theorems 4.1, 4.3, 4.5 so as to be able to explicitly see how they are x -dependent. To do so, we replace the x -dependent path space $D_x \equiv D_x([0, \infty) \rightarrow \mathbf{R}^d) / C_x \equiv C_x([0, \infty) \rightarrow \mathbf{R}^d)$ (with probability measure λ_x / μ_x) by the x -independent path space $D_0 \equiv D_0([0, \infty) \rightarrow \mathbf{R}^d) / C_0 \equiv C_0([0, \infty) \rightarrow \mathbf{R}^d)$ of the paths $X(s) / B(s)$ starting from 0 in space \mathbf{R}^d at time $s = 0$ (with probability measure λ_0 / μ_0), respectively. Namely, in the path integral representation formulas in these three theorems, we make change of space, probability measure and paths by translation x :

$$\begin{aligned} D_x &\rightarrow D_0, \lambda_x \rightarrow \lambda_0, X(s) \rightarrow X(s) + x, \\ C_x &\rightarrow C_0, \mu_x \rightarrow \mu_0, B(s) \rightarrow B(s) + x, B(T(s)) \rightarrow B(T(s)) + x, \end{aligned}$$

$$(4.9) \quad : \quad (e^{-t[H^{(1)} - m]}g)(x) = \int_{D_0([0, \infty) \rightarrow \mathbf{R}^d)} e^{-S^{(1)}(X; x, t)} g(X(t) + x) d\lambda_0(X),$$

$$\begin{aligned} S^{(1)}(X; x, t) &= i \int_0^{t+} \int_{|y| > 0} A(X(s-) + x + \frac{y}{2}) \cdot y \tilde{N}_X(ds dy) \\ &\quad + i \int_0^t \int_{|y| > 0} [A(X(s) + x + \frac{y}{2}) - A(X(s) + x)] \cdot y ds n(dy) \\ &\quad + \int_0^t V(X(s) + x) ds; \end{aligned}$$

$$(4.20) \quad : \quad (e^{-t[H^{(2)} - m]}g)(x) = \int_{D_0([0, \infty) \rightarrow \mathbf{R}^d)} e^{-S^{(2)}(X; x, t)} g(X(t) + x) d\lambda_0(X),$$

$$\begin{aligned} S^{(2)}(X; x, t) &= i \int_0^{t+} \int_{|y| > 0} \left(\int_0^1 A(X(s-) + x + \theta y) d\theta \right) \cdot y \tilde{N}_X(ds dy) \\ &\quad + i \int_0^t \int_{|y| > 0} \left[\int_0^1 A(X(s) + x + \theta y) d\theta - A(X(s)) \right] \cdot y ds n(dy) \\ &\quad + \int_0^t V(X(s) + x) ds; \end{aligned}$$

$$(4.27) \quad : \quad (e^{-t[H^{(3)} - m]}g)(x) = \int_{\substack{C_0([0, \infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0, \infty) \rightarrow \mathbf{R}^d)}} e^{-S^{(3)}(B, T; x, t)} g(B(T(t)) + x) d\mu_0(B) d\nu_0(T),$$

$$\begin{aligned} S^{(3)}(B, T; x, t) &= i \int_0^{T(t)} A(B(s) + x) \cdot dB(s) + \frac{i}{2} \int_0^{T(t)} \operatorname{div} A(B(s) + x) ds \\ &\quad + \int_0^t V(B(T(s)) + x) ds, \\ &\equiv i \int_0^{T(t)} A(B(s) + x) \circ dB(s) + \int_0^t V(B(T(s)) + x) ds \end{aligned}$$

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